Spectrum of periodic signals

Signals and codes (SK)

Department of Transport Telematics
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Lecture 3

Lecture goal and content

Goal

- Be able to find spectral representation of arbitrary periodical signal and reconstruct the signal back from the spectrum.

Content

- Multiplication of sinusoids – beat note
- Amplitude modulation (AM) principle
- Periodic and nonperiodic signals
- Fourier series
- Fourier analysis
- Fourier synthesis
- From Fourier series to Fourier transform
- From Fourier transform to Short time Fourier transform

SPECTRUM OF PERIODIC SIGNALS

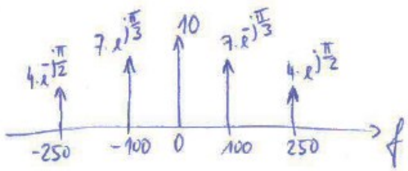
Ex.3_1 Find two-sided spectrum (recall the last lecture) of $x(t) = 10 + 14\cos(200\pi t - \frac{\pi}{3}) + 8\cos(500\pi t + \frac{\pi}{2})$

Sol.: $x(t) = 10 + \frac{14}{2}\left(e^{j(200\pi t - \frac{\pi}{3})} + e^{-j(200\pi t - \frac{\pi}{3})}\right) + \frac{8}{2}\left(e^{j(500\pi t + \frac{\pi}{2})} + e^{-j(500\pi t + \frac{\pi}{2})}\right) =$

$= 10 + 7 \cdot e^{-j\frac{\pi}{3}} \cdot e^{j2\pi \cdot 100t} + 7 \cdot e^{j\frac{\pi}{3}} \cdot e^{-j2\pi \cdot 100t} + 4 \cdot e^{j\frac{\pi}{2}} e^{j2\pi \cdot 250t} + 4 \cdot e^{-j\frac{\pi}{2}} \cdot e^{-j2\pi 250t}$

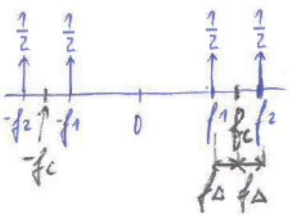
$\rightsquigarrow$ Spectrum is $\{(0,10), (100, 7e^{-j\frac{\pi}{3}}), (-100, 7e^{j\frac{\pi}{3}}), (250, 4e^{j\frac{\pi}{2}}), (-250, 4e^{-j\frac{\pi}{2}})\}$

Spectrum plot:



Multiplication of sinusoids - beat note

Consider $x(t) = \cos(2\pi f_1 t) + \cos(2\pi f_2 t)$ $\rightsquigarrow$ spectrum



Denote $f_1 = f_c - f_\Delta$, $f_2 = f_c + f_\Delta$

center frequency

frequency deviation

Then $f_c = \frac{1}{2}(f_1 + f_2)$, $f_\Delta = \frac{1}{2}(f_2 - f_1)$

$x(t) = \cos(2\pi f_1 t) + \cos(2\pi f_2 t) = \mathrm{Re}\{e^{j2\pi f_1 t}\} + \mathrm{Re}\{e^{j2\pi f_2 t}\} =$

$= \mathrm{Re}\{e^{j2\pi(f_c - f_\Delta)t}\} + \mathrm{Re}\{e^{j2\pi(f_c + f_\Delta)t}\} = \mathrm{Re}\{e^{j2\pi f_c t}(e^{-j2\pi f_\Delta t} + e^{j2\pi f_\Delta t})\} =$

$= \mathrm{Re}\{e^{j2\pi f_c t} \cdot 2\cos(2\pi f_\Delta t)\} = 2 \cdot \cos(2\pi f_c t) \cdot \cos(2\pi f_\Delta t)$

multiplication leads to addition and vice versa

03_1

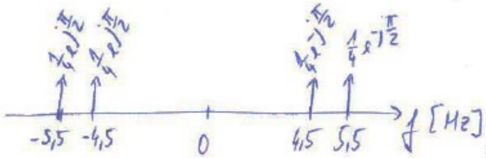
Ex.3_2: Find spectrum of a product $x(t) = \cos\pi t \cdot \sin 10\pi t$

Sol.: $x(t) = \cos\pi t \cdot \sin 10\pi t = \frac{e^{j\pi t} + e^{-j\pi t}}{2} \cdot \frac{e^{j10\pi t} - e^{-j10\pi t}}{2j} =$

$= \frac{e^{j\pi t} + e^{-j\pi t}}{2} \cdot \frac{e^{j10\pi t} \cdot e^{-j\frac{\pi}{2}} + e^{-j10\pi t} e^{j\frac{\pi}{2}}}{2} =$

$= \frac{1}{4}\left(e^{j11\pi t} \cdot e^{-j\frac{\pi}{2}} + e^{-j9\pi t} \cdot e^{j\frac{\pi}{2}} + e^{j9\pi t} \cdot e^{-j\frac{\pi}{2}} + e^{-j11\pi t} \cdot e^{j\frac{\pi}{2}}\right) = \frac{1}{2}\cos(11\pi t - \frac{\pi}{2}) + \frac{1}{2}\cos(9\pi t - \frac{\pi}{2}) = \frac{1}{2}(\sin 11\pi t + \sin 9\pi t)$

Not necessary for this Ex.



note: original frequencies 0,5 Hz and 5 Hz are missing in the spectrum, but there is $5 \pm 0{,}5$ Hz components instead.

f_c coming from $\sin 10\pi t$ $\quad$ f_Δ coming from $\cos\pi t$

Amplitude modulation (AM) principle

$v(t)$ ... transmitted signal (e.g. voice)

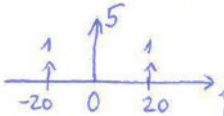
$\cos 2\pi f_c t$ ... carrier signal, f_c ... carrier frequency

AM signal: $x(t) = v(t) \cdot \cos(2\pi f_c t)$

nearly the same as the beat note, but including f_c frequency component

Ex.3_3: Derive the spectrum of $v(t) = 5 + 2\cos 2\pi 20t$ and plot it.

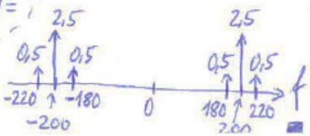
Sol.: $v(t) = 5 + e^{j2\pi 20t} + e^{-j2\pi 20t}$



Ex.3_4: Now take $v(t)$ from Ex.3_3 and derive the spectrum for $x(t) = v(t) \cdot \cos(2\pi 200t)$ and plot it.

Sol.: $x(t) = \left(5 + e^{j2\pi 20t} + e^{-j2\pi 20t}\right) \cdot \frac{1}{2}\left(e^{j2\pi 200t} + e^{-j2\pi 200t}\right) =$

$= \frac{1}{2}\left(5e^{j2\pi 200t} + 5e^{-j2\pi 200t} + e^{j2\pi 220t} + e^{-j2\pi 180t} + e^{j2\pi 180t} + e^{-j2\pi 220t}\right)$



note: compare spectrums in last two exercises

note: Real AM broadcasting … 150 kHz < f_c < 26 MHz, channel spacing 9 kHz
(broad band) (base band)

Periodic and nonperiodic signals

- **Periodic signals** $x(t) = x(t+T_0)$, T_0 … fundamental period

Such signals can be reconstructed from cosines of harmonically related frequencies, i.e. if ALL frequencies are integer multiple of f_0, signal can be reconstructed using N+1 waveforms:

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k) \quad \text{, where}$$

f_0 … fundamental frequency, greatest common divisor (g.c.d.) of all the signal frequencies

f_k … k^{th} harmonic frequency

Ex. 3.5: Signal contains frequencies $\{1{,}2; 2; 6\}$ kHz. Find the fundamental frequency, state which harmonic frequencies does the signal contain. Is the signal periodic?

Sol.: $g.c.d.(1{,}2; 2; 6) = \underline{0{,}4 \text{ kHz} = f_0}$

$\frac{1{,}2}{0{,}4} = 3^{th}$ harmonic, 5^{th}, 15^{th} Signal is periodic

- **Nonperiodic signals**

What if one frequency in a signal exists, which is not a rational multiple of each other?

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$ … still valid, BUT no assumptions about f_k

Periodicity is tied to harmonic frequencies!

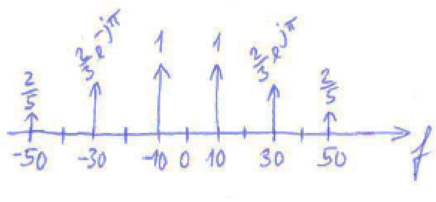
Ex. 3.6: Consider two signals which seem nearly the same:

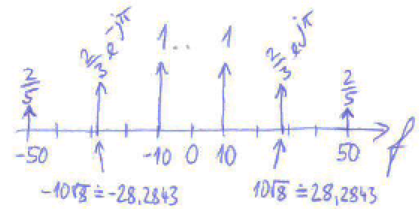
$$x_1(t) = 2\cos(20\pi t) - \frac{2}{3}\cos(20\pi \cdot 3 \cdot t) + \frac{2}{5}\cos(20\pi \cdot 5 \cdot t)$$

$$x_2(t) = 2\cos(20\pi t) - \frac{2}{3}\cos(20\pi \cdot \sqrt{8} \cdot t) + \frac{2}{5}\cos(20\pi \cdot 5 \cdot t)$$

Sketch the spectrum plot of these signal, plot them in duration of 10 periods using SW of your choice. Observe, which one is periodic.

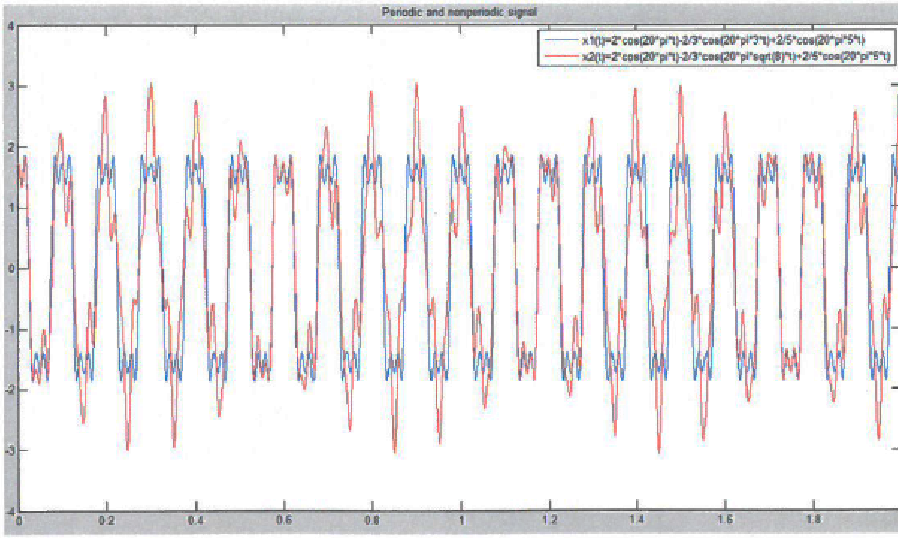
Sol.: Spectrum of $x_1(t)$ Spectrum of $x_2(t)$





$f_0 = 10$ Hz $\leadsto$ has to be periodic

$f_0 \to 0$ … can not be periodic



03_2

Fourier series

- above examples show, that we can synthesize PERIODIC signals by suming harmonically related sinusoids.

Fourier series build general theory; how any periodic signal can be synthesized using sum of harmonically related sinusoids.

Fourier series: $\boxed{x(t)=\sum_{k=-\infty}^{+\infty} a_k \cdot e^{j2\pi f_0 k t}}$, where

f_0 ... fundamental frequency, $f_0=\frac{1}{T_0}$ ← fundamental period

$f_k=k\cdot f_0$ ... k^{th} harmonic frequency

2 aspects of Fourier theory:

- Fourier analysis ... starting from $x(t)$ ... calculating set of $\{a_k\}$
- Fourier synthesis ... starting from $\{a_k\}$ ... calculating $x(t)$

Q.: What is the relation between $\{a_k\}$ and spectrum of the signal $x(t)$?

A.: $\{a_k\}$ is set of complex amplitudes a_k. Each a_k is exactly a value of spectral line at respective frequency f_k!

Fourier analysis

How to calculate complex amplitudes a_k?

By Fourier integral $$a_k=\frac{1}{T_0}\int_0^{T_0} x(t)\cdot e^{-j2\pi f_0 k t}\,dt$$

note: integral limits are arbitrary, when covering one whole period T_0

note: for $k=0$ we obtain $a_0=\frac{1}{T_0}\int_0^{T_0} x(t)\,dt$, which is already known as an average value (or DC value) of periodic signal.

Fourier synthesis

How to calculate $x(t)$ from set of $\{a_k\}$?

We use original Fourier series formula $x(t)=\sum_{k=-\infty}^{+\infty} a_k \cdot e^{j2\pi f_0 k t}$

Then we use inverse Euler formulas and due to symetry $a_k=a^*_{+k}$ we finally obtain some combination of cosine functions.

Ex. 3.7: Perform Fourier analysis of a signal $x(t)=\cos(2\pi f_0 t+\frac{\pi}{8})$

Sol.: $a_k=\frac{1}{T_0}\int_0^{T_0} x(t)\cdot e^{-j2\pi f_0 k t}\,dt=\frac{1}{T_0}\int_0^{T_0}\cos(2\pi f_0 t+\frac{\pi}{8})\cdot e^{-j2\pi f_0 k t}\,dt=$

$=\frac{1}{T_0}\int_0^{T_0}\frac{1}{2}\left(e^{j(2\pi f_0 t+\frac{\pi}{8})}+e^{-j(2\pi f_0 t+\frac{\pi}{8})}\right)\cdot e^{-j2\pi f_0 k t}\,dt=$

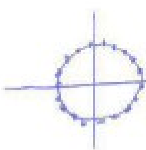
$=\frac{1}{2T_0}\int_0^{T_0} e^{j\frac{\pi}{8}}\cdot e^{j2\pi f_0 t(1-k)}+e^{-j\frac{\pi}{8}}\cdot e^{-j2\pi f_0 t(1+k)}\,dt$

note: k represents all integers

For $k=1$: $a_1=\frac{1}{2T_0}e^{j\frac{\pi}{8}}\int_0^{T_0} e^0\,dt=\frac{1}{2}e^{j\frac{\pi}{8}}$

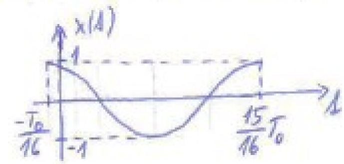
For $k=-1$: $a_{-1}=\frac{1}{2}e^{-j\frac{\pi}{8}}$

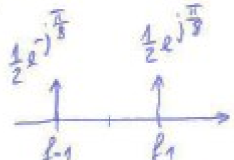
For all other k we obtain $a_k=0$, because $\int_0^{T_0} e^{j2\pi f_0 \ell t}=0$ for all integer $\ell\neq 0$.



sum of all numbers on a circle is equal to zero

Solution is done. We can plot original signal and the spectrum, i.e. $\{a_k\}$.





03_3

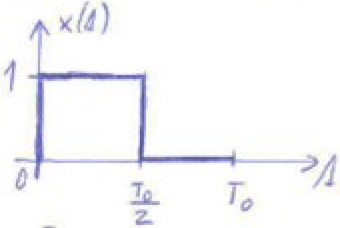
notes: • The spectrum is as expected, but we observed the procedure of computing Fourier integral.

• We observed so called orthogonality property of complex exponential, i.e. from the „long" expression inside the Fourier integral for the specific $\underline{k} = \underline{l}$ only the summand with

$$e^{j2\pi f_0 t(\underline{l}-\underline{k})} = e^{j2\pi f_0 t\cdot 0} = e^{j\cdot 0} = 1$$

leads to nonzero result of the integral.

Ex. 3_8 : Find spectrum of graphically given continuous time signal $x(t)$ with a fundamental period T_0.



Sol.:

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)\cdot e^{-j2\pi f_0 k t}\,dt = \frac{1}{T_0}\int_0^{\frac{T_0}{2}} 1\cdot e^{-j2\pi f_0 k t}\,dt =$$

$$= \frac{1}{T_0}\cdot\frac{1}{-j2\pi f_0 k}\left[e^{-j2\pi f_0 k t}\right]_0^{\frac{T_0}{2}} = \frac{1}{-j2\pi k}\left(e^{-j2\pi f_0 k \frac{T_0}{2}} - 1\right) =$$

$$T_0 = \frac{1}{f_0}$$

$$= \frac{1}{-j2\pi k}\left(e^{-j\pi k} - 1\right)$$

→ for even $\underline{k}$ ... $e^{-j\pi k} = 1 \leadsto \underline{a_k = 0}$ (except for k=0)

→ for odd $\underline{k}$ ... $e^{-j\pi k} = -1 \leadsto \underline{a_k} = \frac{1}{-j2\pi k}(-1-1) =$

$$= \frac{1}{j\pi k} = \underline{\frac{1}{k\pi}\cdot e^{-j\frac{\pi}{2}}}$$

So: $a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt = 0.5$

$a_1 = \frac{1}{\pi}e^{-j\frac{\pi}{2}}$, $a_{-1} = \frac{1}{-\pi}\cdot e^{-j\frac{\pi}{2}} = \frac{1}{\pi}\cdot e^{j\frac{\pi}{2}}$

$a_2 = 0$, $a_{-2} = 0$

$a_3 = \frac{1}{3\pi}e^{-j\frac{\pi}{2}}$, $a_{-3} = \frac{1}{3\pi}e^{j\frac{\pi}{2}}$

$a_4 = 0$, $a_{-4} = 0$

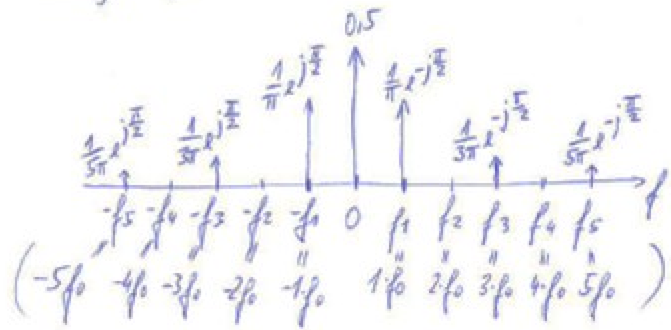
$a_5 = \frac{1}{5\pi}e^{-j\frac{\pi}{2}}$, $a_{-5} = \frac{1}{5\pi}e^{j\frac{\pi}{2}}$

$a_6 = 0$, $a_{-6} = 0$

⋮ etc. ⋮

03-4

Ploting the first 5 harmonics we obtain



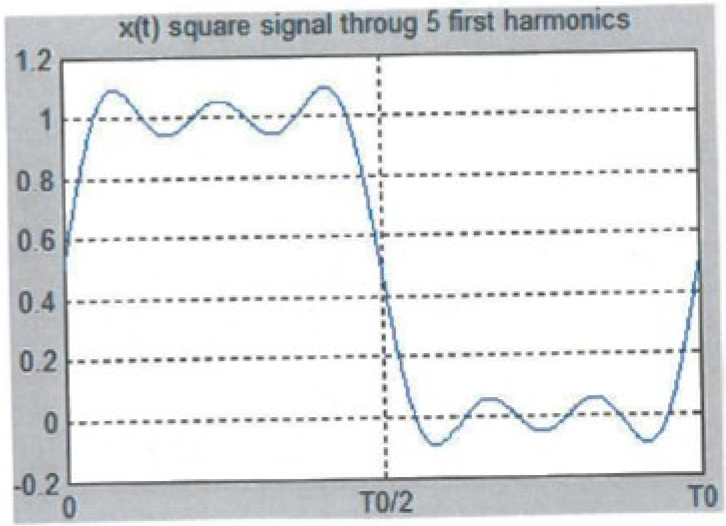
Ex. 3_9 : Synthetise the signal from the resulting spectrum of the square signal (Ex. 3-8) through first 5 harmonics. Plot the result using SW of your choice.

Sol.:

$$x(t) = \sum_{k=-5}^{+5} a_k\cdot e^{j2\pi f_0 k t} = \frac{1}{5\pi}e^{j\frac{\pi}{2}}\cdot e^{j2\pi f_0(-5)t} + \frac{1}{3\pi}e^{j\frac{\pi}{2}}\cdot e^{j2\pi f_0(-3)t} +$$

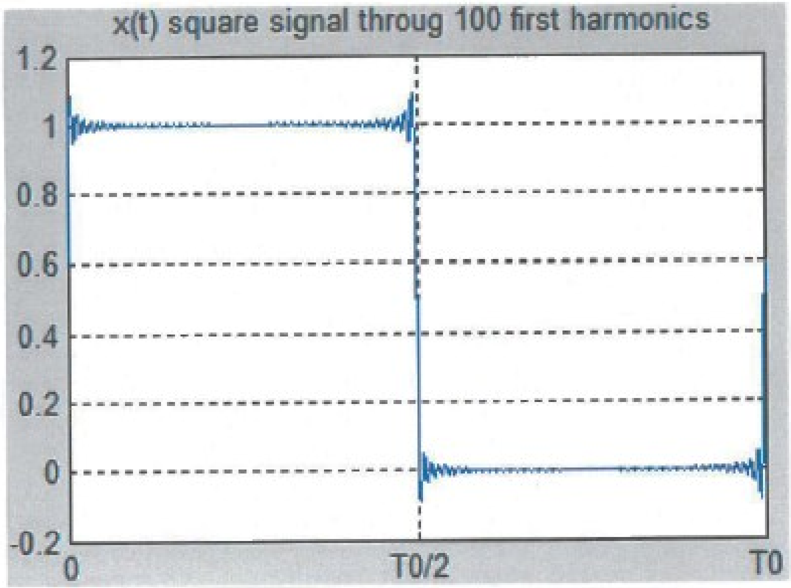
$$+ \frac{1}{\pi}e^{j\frac{\pi}{2}}\cdot e^{j2\pi f_0(-1)t} + 0.5 + \frac{1}{\pi}e^{-j\frac{\pi}{2}}\cdot e^{j2\pi f_0\cdot 1\cdot t} + \frac{1}{3\pi}e^{-j\frac{\pi}{2}}\cdot e^{j2\pi f_0\cdot 3t} + \frac{1}{5\pi}e^{-j\frac{\pi}{2}}\cdot e^{j2\pi f_0\cdot 5t} =$$

$$= 0.5 + \frac{2}{\pi}\cos\left(2\pi f_0 t - \frac{\pi}{2}\right) + \frac{2}{3\pi}\cos\left(2\pi f_0\cdot 3t - \frac{\pi}{2}\right) + \frac{2}{5\pi}\cos\left(2\pi f_0\cdot 5t - \frac{\pi}{2}\right).$$



Ex. 3-10 : The same as Ex. 3-9, but use first 100 harmonics. Just depict a plot of $x(t)$. You can observe so called Gibbs phenomenon, which occurs when there is some discontinuity in the signal.

Sol. :



From Fourier series to Fourier transform

- within Fourier analysis we compute $a_k = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-j2\pi f_0 k t}\, dt =$

$$= \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{2T_0}\int_{-T_0}^{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

How to obtain Fourier transform formula?

- let $T_0 \to \infty$ (signal need not to be periodic)
- compute not a_k, but $a_k \cdot 2 \cdot T_0$ (or simply don't consider $\frac{1}{2T_0}$)
- compute not for discrete radian frequencies $k\omega_0$, but for all ω.

Then we get Fourier transform $\hat{F}(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t}\, dt$

- problematic for power signals, some ω gives $F(\omega) \to \infty$

From Fourier transform to Short time Fourier transform (STFT)

- we multiply the signal with window function (some impulse) $w(t)$
- we obtain „picture" known as spectrogram (see below)

Continuous time: $STFT(\overset{time}{\tau}, \omega) = \int_{-\infty}^{\infty} x(t) \cdot w(t-\tau)\, e^{-j\omega t}\, dt$ window is shifting through all times

Discrete time: $STFT[m, \omega] = \sum_{n=-\infty}^{\infty} x[n]\, w[n-m] \cdot e^{-j\omega n}$

- Examples of window functions

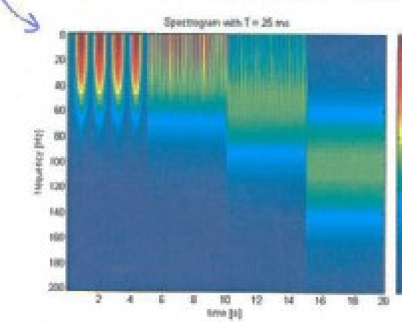
1) Rectangular

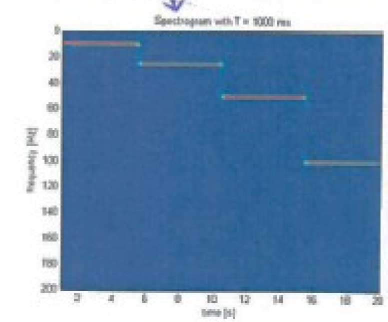
N... window width in samples

2) Triangular

3) Hanning window $w[n] = 0{,}5 \cdot \left(1 - \cos\frac{2\pi n}{N-1}\right)$

- Resolution issues
  - short window → good resolution of time, bad resolution of frequency
  - long window → bad resolution of time, good resolution of frequency

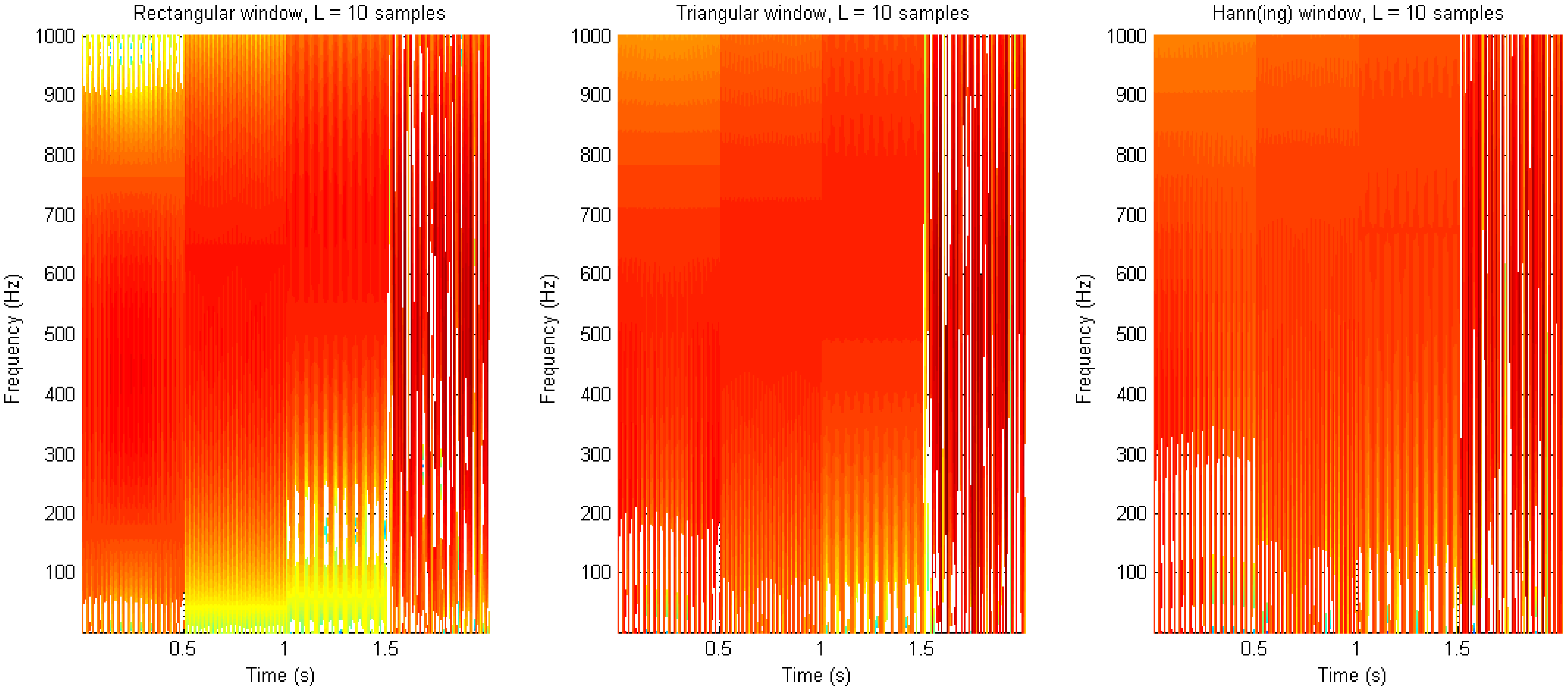


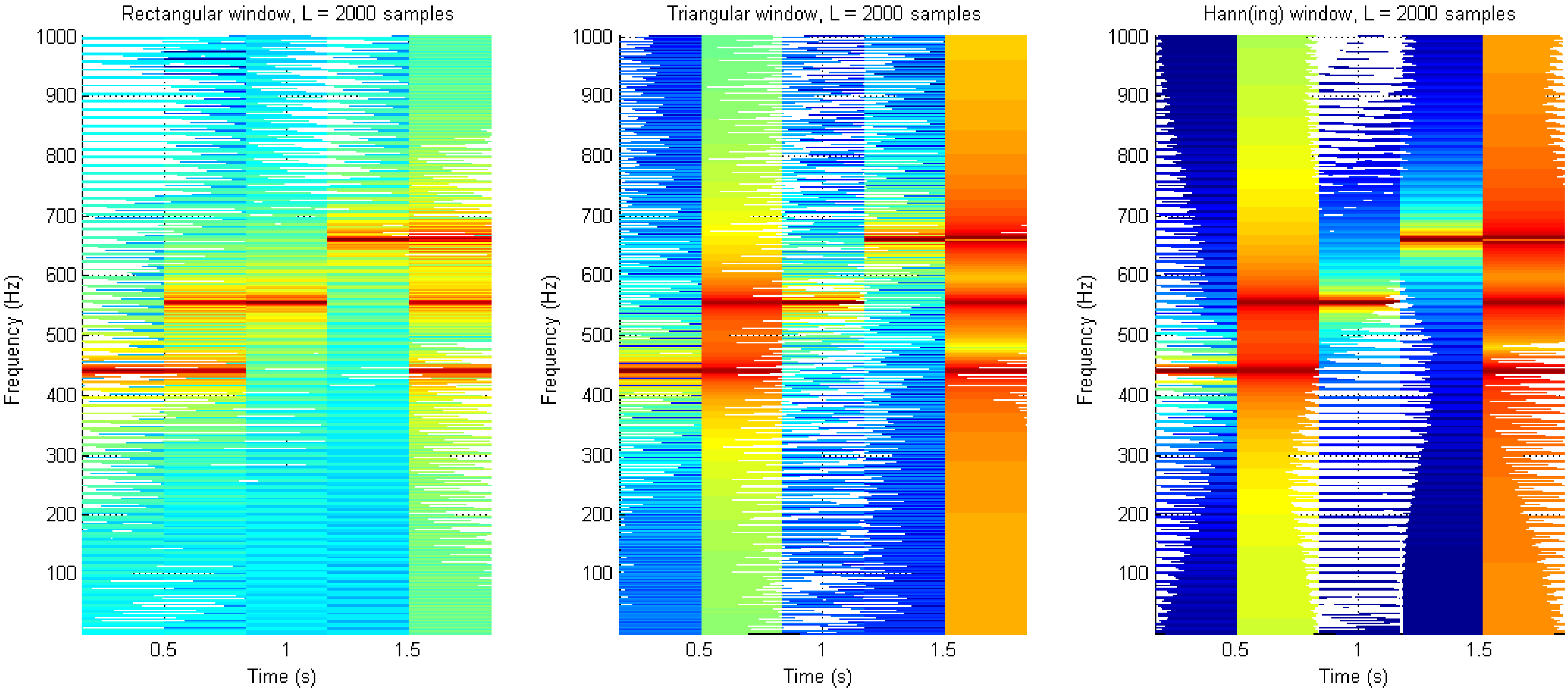


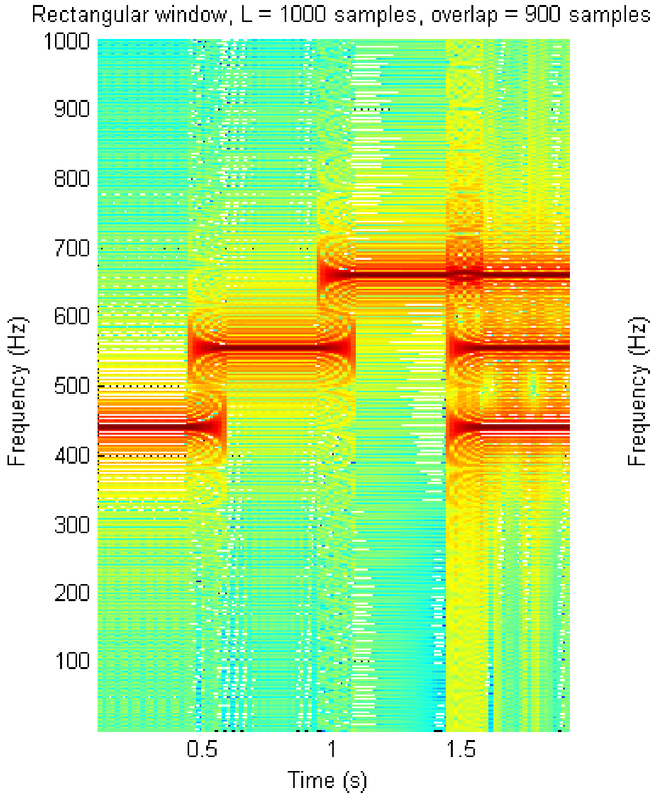
03.5

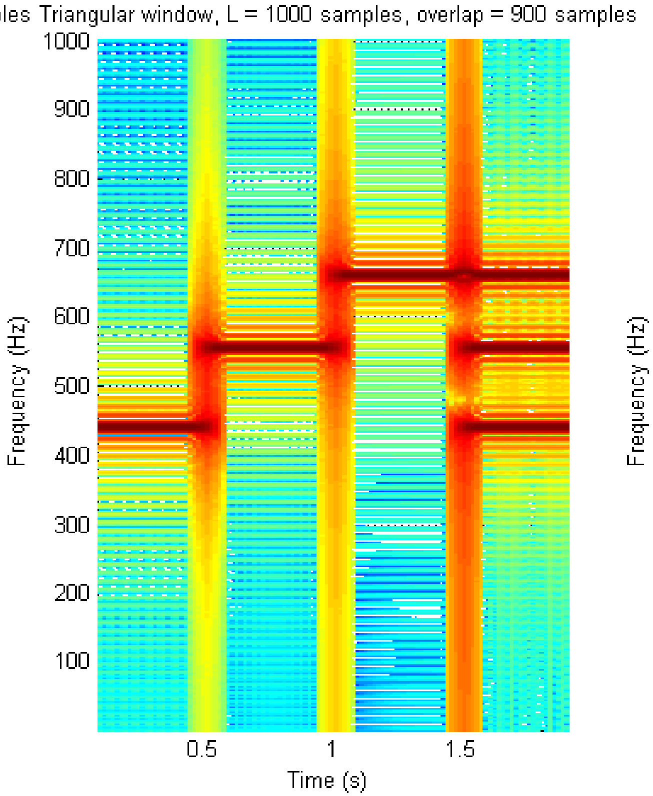
Example of STFT in Matlab – chord A Major

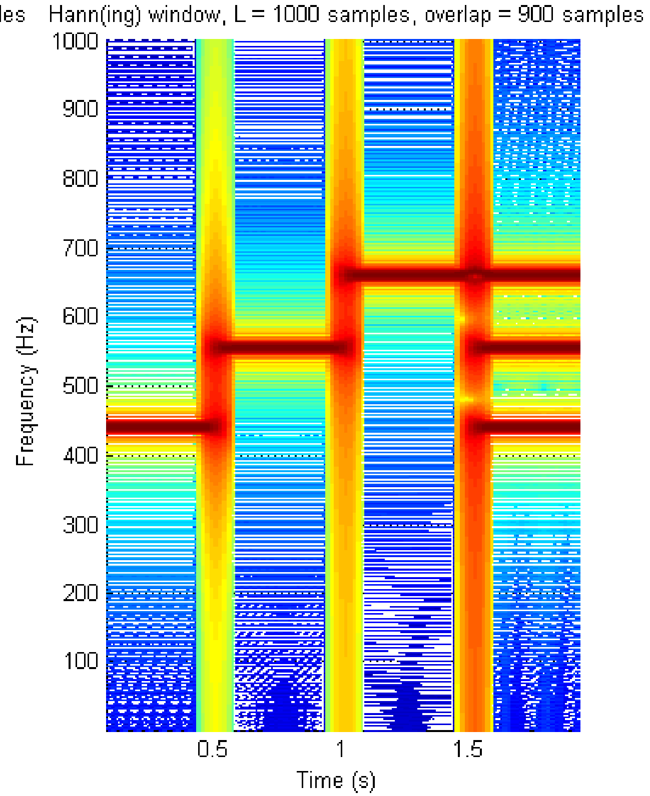
```
% spectrograms with different Windows

%% initialize

clear;
close all;

%% defining parameters

fs=6000; % sample frequency
tdur=0.5; %time duration of one tone in seconds
L=2000; %windows length
noverlap=0; %overlap samples

%% computation
Ts=1/fs;
t_tone=Ts:Ts:tdur;

x1=sin(2*pi*440*t_tone); %tone A
x2=sin(2*pi*440*2^(4/12)*t_tone); %tone C# (Cis)
x3=sin(2*pi*440*2^(7/12)*t_tone); %tone E
x4=x1+x2+x3; %chord A Major (composed of tones A, C# and E), CZ: akord A Dur
x=[x1 x2 x3 x4];
sound(x,fs);
FigHandle = figure('Position', [100, 100, 1300, 500]); %defining position of corners of the figure

subplot(1,3,1)
spectrogram(x,rectwin(L),noverlap,1:1000,fs,'yaxis');
title(sprintf('Rectangular window, L = %d samples',L))
subplot(1,3,2)
spectrogram(x,triang(L),noverlap,1:1000,fs,'yaxis')
title(sprintf('Triangular window, L = %d samples',L))
subplot(1,3,3)
spectrogram(x,hann(L),noverlap,1:1000,fs,'yaxis')
title(sprintf('Hann(ing) window, L = %d samples',L))
```


Rectangular window, L = 10 samples
Triangular window, L = 10 samples
Hann(ing) window, L = 10 samples
Frequency (Hz)
Time (s)
1000
900
800
700
600
500
400
300
200
100
0.5
1
1.5


Rectangular window, L = 2000 samples
Triangular window, L = 2000 samples
Hann(ing) window, L = 2000 samples
Frequency (Hz)
Time (s)
1000
900
800
700
600
500
400
300
200
100
0.5
1
1.5


Rectangular window, L = 1000 samples, overlap = 900 samples
1000
900
800
700
600
500
400
300
200
100
Frequency (Hz)
0.5
1
1.5
Time (s)


Triangular window, L = 1000 samples, overlap = 900 samples
1000
900
800
700
600
500
400
300
200
100
Frequency (Hz)
0.5
1
1.5
Time (s)


Hann(ing) window, L = 1000 samples, overlap = 900 samples
1000
900
800
700
600
500
400
300
200
100
Frequency (Hz)
0.5
1
1.5
Time (s)

Vocabulary EN/CZ

Beat note	Zázněj
Carrier frequency	Nosná frekvence
Greatest common divisor (g.c.d.)	Největší společný dělitel
Center frequency	Střední frekvence
Frequency deviation	Odchylka frekvence
Integer	Celé číslo
Base band	Základní pásmo
Broad band	Přeložené pásmo
Channel spacing	Odstup kanálů
Even	Sudý
Odd	Lichý
Fourier series	Fourierova řada
Window function	Jádro(vá funkce), okno

References

- McClellan, Schafer and Yoder, Signal Processing First, ISBN 0-13-065562-7., Prentice Hall, Upper Saddle River, NJ 07458. 2003 Pearson Education, Inc.