Additional exercises

Signals and codes (SK)

Department of Transport Telematics
Faculty of Transportation Sciences, CTU in Prague

Lecture 6

Lecture goal and content

Goal

- Practice topics up to now by computing several computing exercises.

Content

- Sinusoids
- Spectrum
- Sampling and aliasing

Exercise 1 (sinusoids): add 2 complex exponentials

Problem: For given complex exponential signal $x[n] = e^{j(0,4\pi n - 0,6\pi)}$ express a new signal $y[n] = x[n] - x[n-1]$ in the form $y[n] = Ae^{j(\omega_0 n + \Phi)}$.

Solution:

$$y[m] = e^{j(0,4\pi m - 0,6\pi)} - e^{j(0,4\pi m - 1\pi)} = e^{j0,4\pi m}\left(e^{-j0,6\pi} - e^{-j\pi}\right) =$$

$$= \left| e^{-j0,6\pi} = -0,309 - 0,951j \right| = e^{j0,4\pi m}\left(-0,309 - 0,951j + 1\right) =$$

$$= e^{j0,4\pi m}\left(0,691 - 0,951j\right) = \left| 0,691 - 0,951j = 1,176 \cdot e^{-0,9425} = 1,176 \cdot e^{-j0,3\pi} \right| =$$

$$= e^{j0,4\pi m} \cdot 1,176 \cdot e^{-j0,3\pi} = 1,176\, e^{j(0,4\pi m - 0,3\pi)}$$

$$\leadsto \underline{A = 1,176}, \quad \underline{\omega_0 = 0,4\pi \tfrac{rad}{s}}, \quad \underline{\phi = -0,3\pi\ rad}$$

Exercise 2 (sinusoids): sinusoid is a sum of 2 rotating phasors

Problem: Let $x(t)$ be the signal $x(t) = -6\sin(400\pi t - 0{,}25\pi)$

a) Express $x(t)$ as a sum of two complex rotating phasors rotating in opposite directions (clockwise and counter-clockwise). Use inverse Euler's formula.

b) Determine complex phasor Z and radian frequency ω such that $x(t) = Re\{Ze^{j\omega t}\}$

Solution:

a) $$x(t) = -6\cdot\frac{e^{j(400\pi t-\frac{\pi}{4})} - e^{-j(400\pi t-\frac{\pi}{4})}}{2j} = \underset{j=e^{j\frac{\pi}{2}}}{3j}\left(e^{j400\pi t}\cdot e^{-j\frac{\pi}{4}} - e^{-j400\pi t}\cdot e^{j\frac{\pi}{4}}\right) =$$

$$= 3\,e^{j400\pi t}\cdot e^{j\frac{\pi}{4}} - 3\,e^{-j400\pi t}\cdot e^{j\frac{3}{4}\pi} = \underbrace{3e^{j400\pi t}\cdot e^{j\frac{\pi}{4}}}_{\text{counter-clockwise}} + \underbrace{3e^{-j400\pi t}\cdot e^{-j\frac{\pi}{4}}}_{\text{clockwise}}$$

b) $$x(t) = -6\sin(\overbrace{400\pi}^{\omega} t - \tfrac{\pi}{4}) = -6\cos(400\pi t - \tfrac{\pi}{4} - \tfrac{\pi}{2}) = 6\cos(400\pi t + \tfrac{\pi}{4})$$

$$6e^{j(400\pi t+\frac{\pi}{4})} = 6\cdot\Big(\underbrace{\cos(400\pi t+\tfrac{\pi}{4})}_{Re} + j\underbrace{\sin(400\pi t+\tfrac{\pi}{4})}_{Im}\Big)$$

$\rightsquigarrow$ $\underline{\underline{Z = 6\cdot e^{j\frac{\pi}{4}}}}$ fulfills the assignment $x(t) = Re\{Z e^{j\omega t}\}$.

Exercise 3 (spectrum): Fourier series coefficients of a sum of sinusoids

Problem: Consider periodic signal $x(t) = 2 + 4\cos(600\pi t + 0{,}25\pi) + \sin(1000\pi t)$

a) Find the period of $x(t)$.

b) Find the Fourier series coefficients of $x(t)$ for $-10 \le k \le 10$.

Solution:

a) $f_0 = g.c.d.(300, 500) = 100\ \text{Hz} \leadsto \underline{\underline{T_0}} = \frac{1}{f_0} = \underline{\underline{10\ ms}}$, $\omega_0 = 2\pi f_0 = 200\pi$

b) $k=0$: $a_k = \underline{\underline{a_0}} = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt = \frac{1}{T_0}\int_0^{T_0} 2\,dt = \underline{\underline{2}}$

For other k using inverse Euler's formula:

$k=3$: $\underline{\underline{a_3 = 2\cdot e^{j\frac{\pi}{4}}}}$, $k=-3$: $\underline{\underline{a_{-3} = 2\cdot e^{-j\frac{\pi}{4}}}}$ $\left(\leadsto 4\cos(600\pi t + \frac{\pi}{4}) = \frac{4e^{j600\pi t}\cdot e^{j\frac{\pi}{4}} + 4e^{-j600\pi t}\cdot e^{-j\frac{\pi}{4}}}{2}\right)$

$k=5$: $\underline{\underline{a_5 = \frac{1}{2}e^{-j\frac{\pi}{2}}}}$, $k=-5$: $\underline{\underline{a_{-5} = \frac{1}{2}e^{j\frac{\pi}{2}}}}$ $\left(\leadsto \sin(1000\pi t) = \cos(1000\pi t - \frac{\pi}{2}) = \frac{1\cdot e^{j1000\pi t}\cdot e^{-j\frac{\pi}{2}} + 1\cdot e^{-j1000\pi t}\cdot e^{j\frac{\pi}{2}}}{2}\right)$

$\underline{\underline{a_k = 0}}$ for all other k

Extra: Try to find a_k coefficients using Fourier integral $a_k = \frac{1}{T_0}\int_0^{T_0} x(t)\,e^{-jk\omega_0 t}\,dt$

Hint: use expression of cosines using inverse Euler's formula.

Exercise 4 (spectrum): Fourier series coefficients of a periodic signal

Problem: Consider signal $x(t)$ periodic with $T_0 = 6$ s defined by the equation $x(t) = \begin{cases} 0 \;\ldots\; 0 \le t < 3 \\ -10 \;\ldots\; 3 \le t < 6 \end{cases}$

a) Sketch the signal $x(t)$ for $-6 \le t \le 12$ s.

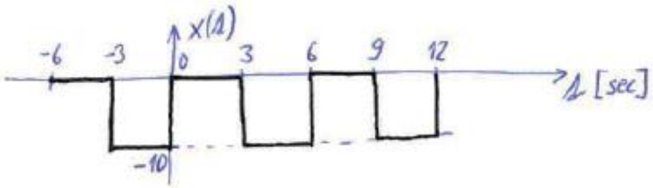
b) Determine average value $\bar{x}$, which is also equal to Fourier series coefficient a_0.

c) Find the Fourier series coefficients a_1 and a_{-1} using Fourier analysis.

d) Would values a_1 and a_{-1} change in case of adding a constant to the original signal $x(t)$?

Solution:

a)



b) $\bar{x} = a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt = \frac{1}{T_0}\int_{\frac{T_0}{2}}^{T_0} -10\,dt = -5$

c) $a_k = \frac{1}{T_0}\int_0^{T_0} x(t)\cdot e^{-j2\pi f_0 k t}\,dt = \frac{1}{T_0}\int_{\frac{T_0}{2}}^{T_0} -10\cdot e^{-j2\pi f_0 k t}\,dt = \frac{-10}{T_0}\,\frac{-1}{j2\pi f_0 k}\left[e^{-j2\pi f_0 k t}\right]_{\frac{T_0}{2}}^{T_0} =$

$\overset{T_0=\frac{1}{f_0}}{=} \frac{-10j}{2\pi k}\cdot\left(e^{-j2\pi f_0 k T_0} - e^{-j2\pi f_0 k \frac{T_0}{2}}\right) = \frac{-10j}{2\pi k}\left(e^{-j2\pi k} - e^{-j\pi k}\right)$

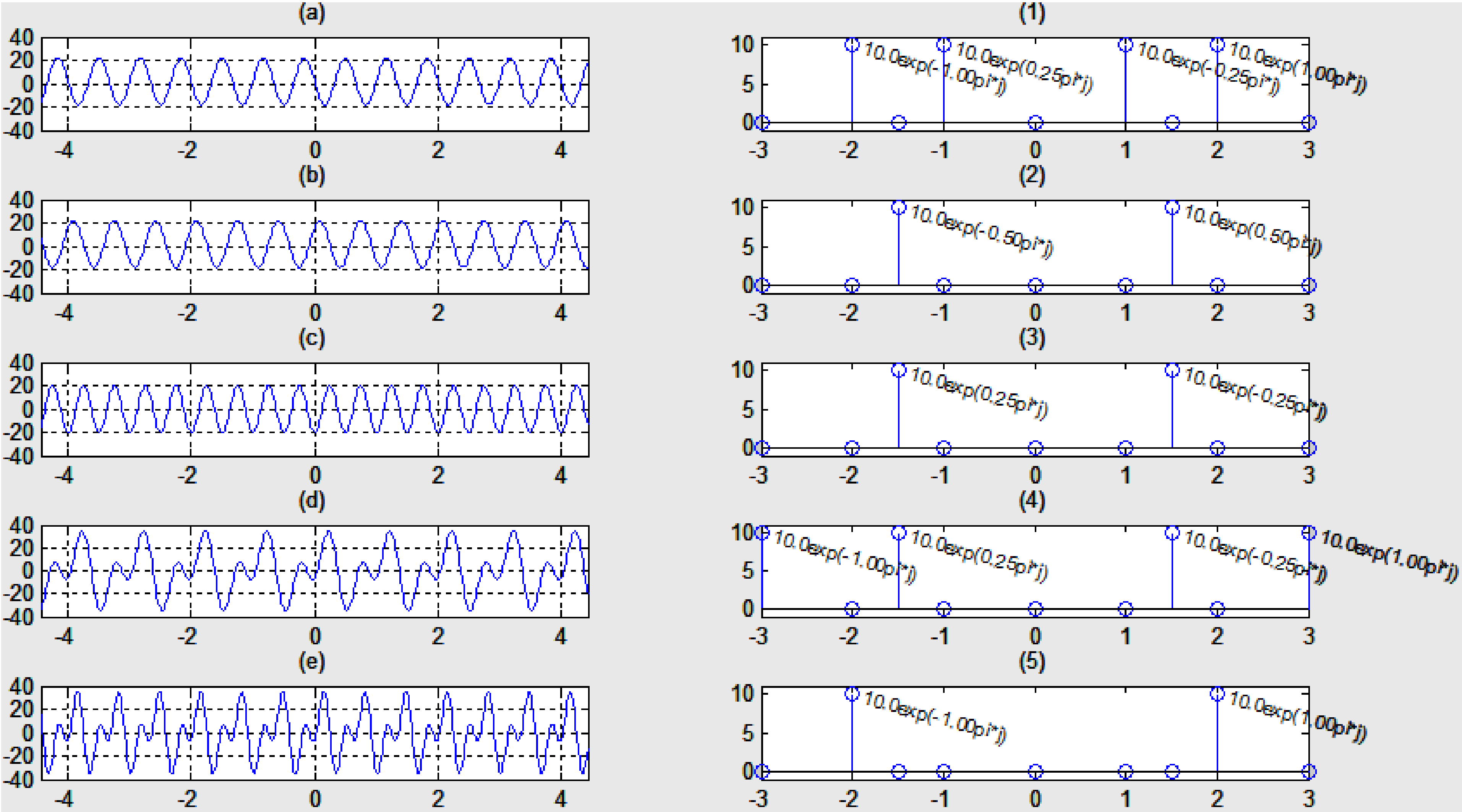
$\Rightarrow$ for $k=1$: $a_1 = \frac{-10j}{2\pi}\left(e^{-j2\pi} - e^{-j\pi}\right) = \frac{-10j}{2\pi}(1-(-1)) = \frac{-10j}{\pi} \doteq 3{,}183\cdot e^{-j\frac{\pi}{2}}$

for $k=-1$: $a_{-1} = \frac{-10j}{-2\pi}\left(e^{j2\pi} - e^{j\pi}\right) = \frac{10j}{2\pi}(1-(-1)) = \frac{10j}{\pi} \doteq 3{,}183\cdot e^{j\frac{\pi}{2}}$

d) No, adding a constant influences just the value of a_0.

Exercise 5 (spectrum): Periodic signals and its spectra

Problem: Assign correct spectrum (1)-(5) to corresponding signal (a)-(b)



Solution: 1d, 2a, 3b, 4e, 5c

Exercise 6 (spectrum): Spectrum of AM signal

Problem: Amplitude modulated signal is expressed by the equation $x(t) = (A + \sin\omega_0 t)\sin\omega_c t$, with $0 < \omega_0 \ll \omega_c$.

a) Use phasors to express $x(t)$ in the form $x(t) = A_1\cos(\omega_1 t + \Phi_1) + A_2\cos(\omega_2 t + \Phi_2) + A_3\cos(\omega_3 t + \Phi_3)$, where $\omega_1 < \omega_2 < \omega_3$. Find values of $A_1, \omega_1, \Phi_1, A_2, \omega_2, \Phi_2, A_3, \omega_3, \Phi_3$ in terms of original parameters A, ω_0, ω_c.

b) Sketch the two-sided spectrum of the signal $x(t)$. Label the plot properly.

Solution: a)

$$x(t) = \left(A + \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right) \cdot \frac{e^{j\omega_c t} - e^{-j\omega_c t}}{2j} =$$

$$+\frac{A}{2} e^{j(-\omega_c t + \frac{\pi}{2})}$$

$$= \frac{A}{2} e^{j(\omega_c t - \frac{\pi}{2})} - \frac{A}{2} e^{j(-\omega_c t - \frac{\pi}{2})} - \frac{1}{4} e^{j(\omega_0 t + \omega_c t)} + \frac{1}{4} e^{j(\omega_0 t - \omega_c t)} + \frac{1}{4} e^{j(\omega_c t - \omega_0 t)} - \frac{1}{4} e^{j(-\omega_0 t - \omega_c t)} =$$

$$= A\cos\left(\omega_c t - \frac{\pi}{2}\right) - \frac{1}{2}\cos(\omega_c + \omega_0)t + \frac{1}{2}\cos(\omega_c - \omega_0)t =$$

$$= \frac{1}{2}\cos(\omega_c - \omega_0) + A\cos\left(\omega_c t - \frac{\pi}{2}\right) + \frac{1}{2}\cos\left[(\omega_c + \omega_0)t + \pi\right]$$

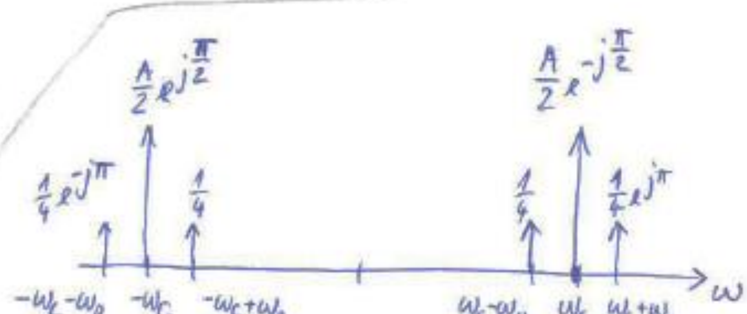
Thus:

$A_1 = \frac{1}{2}$, $\omega_1 = \omega_c - \omega_0$, $\phi_1 = 0$

$A_2 = A$, $\omega_2 = \omega_c$, $\phi_2 = -\frac{\pi}{2}$

$A_3 = \frac{1}{2}$, $\omega_3 = \omega_c + \omega_0$, $\phi_3 = \pi$

b) See spectrum in the plot. $\omega = 2\pi f$



Exercise 7 (spectrum): Spectrum of 2 sinusoids multiple, Matlab code

```
clear; close all;
f0=3; T0=1/f0; om0=2*pi*f0;
fs=1800; Ts=1/fs;
noT=3;
t=0:Ts:noT*T0;
xc=cos(7*om0*t);
xs=sin(om0*t);
x=xc.*xs;
plot(t,x);
hold on
plot(t,xs,'--c',t,-xs,'--c')
```

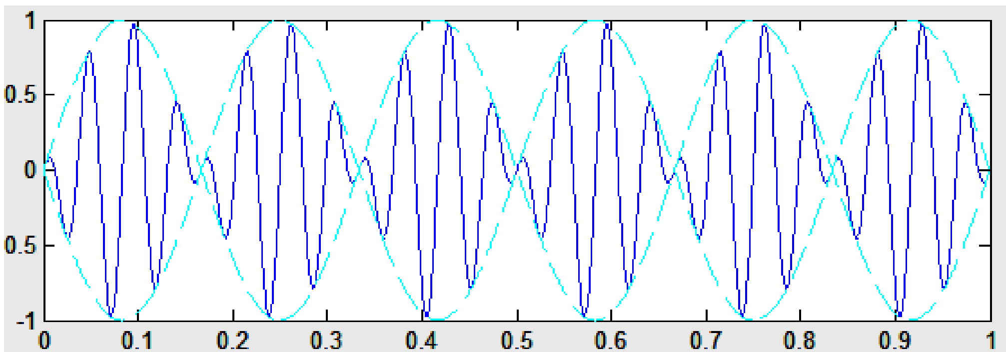
Problem: See the framed Matlab script

a) Sketch and label properly the plot, that would be made by the script.

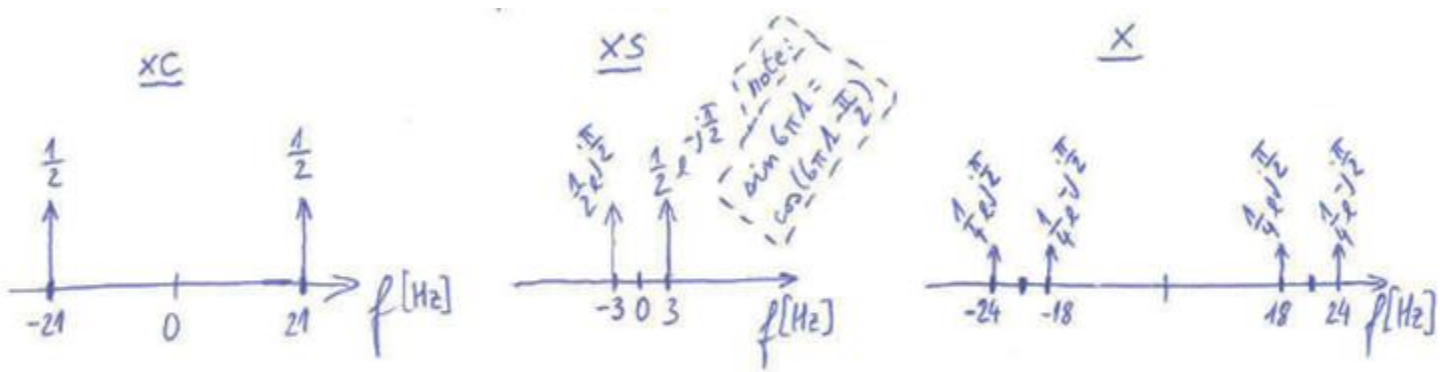
b) Sketch the two-sided spectrum for each of the three signals xc, xs and x.

Solution:

a)



b)



$$x = \frac{e^{j42\pi t} + e^{-j42\pi t}}{2} \cdot \frac{e^{j6\pi t} - e^{-j6\pi t}}{2j} = \frac{1}{4j}\left(e^{j48\pi t} - e^{j36\pi t} + e^{-j36\pi t} - e^{-j48\pi t}\right) = \frac{1}{4}\left(e^{j(48\pi t - \frac{\pi}{2})} + e^{j(36\pi t + \frac{\pi}{2})} + e^{j(-36\pi t - \frac{\pi}{2})} + e^{j(-48\pi t + \frac{\pi}{2})}\right)$$

Problem: Discrete-time signal $x[n] = 325\cos(0{,}35\pi n - \pi/6)$ was obtained by sampling original continuous-time signal at sampling rate $f_s = 2500$ samples/second.

a) Determine formulas for two different continuous-time signals $x_1(t)$ and $x_2(t)$ whose samples are equal to $x[n]$. Both signals shall have a frequency less than 2500 Hz.

Solution:

a) $\hat{\omega}_0 = 0{,}35\pi \qquad \left(\hat{\omega}_0 = 2\pi \frac{f_0}{f_s}\right)$

$\rightsquigarrow$ first possible CT signal: $f_{01} = \frac{\hat{\omega}_0 \cdot f_s}{2\pi} = 437{,}5$ Hz, thus $\underline{x_1(t) = 325\cos(875\pi t - \frac{\pi}{6})}$

second possible CT signal: (1^{st} alias has $\hat{\omega}_{0(1a)} = 0{,}35\pi + 2\pi = 2{,}35\pi \rightsquigarrow f_{02} = \frac{\hat{\omega}_{0(1a)} \cdot f_s}{2\pi} = 5875$ Hz

This is not possible: frequency shall be less than 2500 Hz)

1^{st} folded alias $\hat{\omega}_{0(1f)} = -0{,}35\pi + 2\pi = 1{,}65\pi \rightsquigarrow f_{02} = \frac{\omega_{0(1f)} \cdot f_s}{2\pi} = 2062{,}5$ Hz $(= f_s - f_{01})$

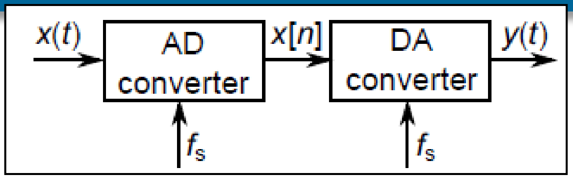
Thus $\underline{x_2(t) = 325\cos(4125\pi t + \frac{\pi}{6})}$

for folding negating sign of phase necessary!

(Deriving: $t = m \cdot T_s = \frac{m}{f_s} \rightsquigarrow x_2(t) = x_2\left(\frac{m}{f_s}\right) = A\cos\left(2\pi \frac{f_s - f_{01}}{f_s} m + \phi\right) = A\cos\left(2\pi m - 2\pi \frac{f_0}{f_s} m + \phi\right) =$

$= A\cos\left(2\pi \frac{f_0}{f_s} m - \phi\right)$ ... that is why ϕ is changing sign)

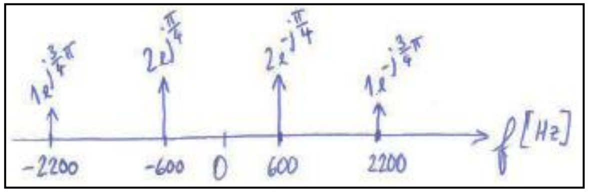
Exercise 9 (sampling): AD and DA converter in cascade, input spectrum



Problem:

Consider a system according to the framed block diagram.

a) Determine $y(t)$, if $x(t)$ is given by two-sided spectrum. Consider $f_s = 1000$ samples/sec



Solution:

a) $x(t) = 4\cos(1200\pi t - \frac{\pi}{4}) + 2\cos(4400\pi t - \frac{3}{4}\pi)$

$f = 600\,Hz$: $\hat{\omega} = 2\pi \frac{600}{1000} = 1{,}2\pi$ ... not within range $-\pi \le \hat{\omega} \le \pi$

$\hat{\omega} = 1{,}2\pi$ is 1st folding alias of what will appear as an output $\hat{\omega} = 0{,}8\pi$

(as $-0{,}8\pi + 1\cdot 2\pi = 1{,}2\pi$) lies within range $-\pi \le \hat{\omega} \le \pi$

$f = 2200\,Hz$: $\hat{\omega} = 2\pi \frac{2200}{1000} = 4{,}4\pi$ ... not within range $-\pi \le \omega \le \pi$

$\hat{\omega} = 4{,}4\pi$ is 2nd alias of what will appear as an output $\hat{\omega} = 0{,}4\pi$

(as $0{,}4\pi + 2\cdot 2\pi = 4{,}4\pi$)

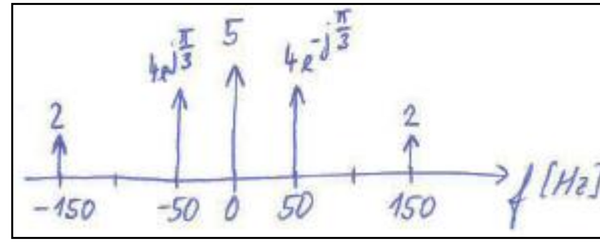
So: $x[n] = 4\cos(0{,}8\pi n + \frac{\pi}{4}) + 2\cos(0{,}4\pi n - \frac{3}{4}\pi)$

for $f = 600\,Hz$ folding $\Rightarrow$ negate phase shift

$t = n\cdot T_s = \frac{n}{f_s}$, $f_s = 1000$, thus $y(t) = 4\cos(800\pi t + \frac{\pi}{4}) + 2\cos(400\pi t - \frac{3}{4}\pi)$

Exercise 10 (sampling): Discrete-time signal from spectrum and sampling

Problem: Consider signal $x(t)$ given by two-sided spectrum



a) Write an equation for $x(t)$

b) Write an equation for $x[n]$, that will originate from $x(t)$ through sampling with $f_s = 150$ Hz

c) Sketch spectrum of $x[n]$ from subtask b)

Solution:

a) $x(t) = 5 + 8\cos\left(100\pi t - \frac{\pi}{3}\right) + 4\cos 300\pi t$

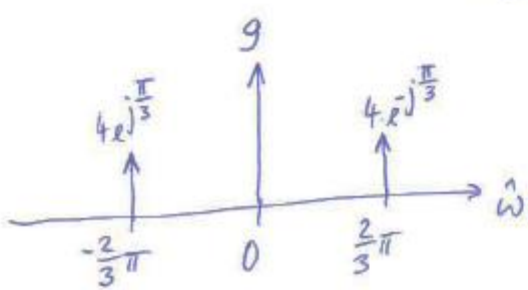
b) $f_0 = 50$ Hz: $\hat{\omega}_0 = 2\pi \cdot \frac{f_0}{f_s} = 2\pi \frac{50}{150} = \frac{2}{3}\pi$ … O.K., within range $-\pi \le \hat{\omega} \le \pi$

$f = 150$ Hz: $\hat{\omega} = 2\pi \cdot \frac{150}{150} = 2\pi$ … outside range $-\pi \le \hat{\omega} \le \pi$.

$\hat{\omega} = 2\pi$ is 1st alias of principal alias 0π (as $0\pi + 1 \cdot 2\pi = 2\pi$)

$x[n] = 5 + 8 \cdot \cos\left(\frac{2}{3}\pi n - \frac{\pi}{3}\right) + 4 \cdot \cos(0\pi n) = 9 + 8\cos\left(\frac{2}{3}\pi n - \frac{\pi}{3}\right)$

c)

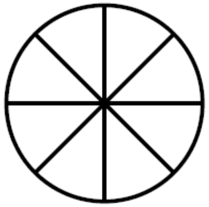


Exercise 11 (sampling): Spoked wheel

Problem: Consider rotating spoked wheel seen in TV with 30 frames/sec sampling used for transmitting TV images. Assume clockwise rotating at a constant speed 4 rev/sec

a) Find continuous-time equation for rotating phasor $p(t)$ which represents observed movement of an individual spoke.

b) Write a formula for $p[n]$, the movement of an individual spoke as a function of frame index n



c) Determine speed and direction of rotating, that a TV viewer will see.

d) Which speed would appear to the TV viewer, that the wheel doesn't move at all.

Solution:

a)

For the outer end of spoke 1 and for radius $r=1$ we get position $p(t) = e^{-j2\pi\cdot 4\cdot t}$

b) $t = n\cdot T_s = \frac{n}{f_s}$ ⇝ $p[n] = e^{-j8\pi\cdot\frac{n}{30}} \doteq e^{-j\pi n\cdot 0{,}267}$

c) There are 8 spokes ⇝ $\Delta\varphi = \frac{2\pi}{8} = 0{,}25\pi$ rad

What happens within 1st frame: spoke 1 rotates by $-0{,}267\pi$ rad, but it looks like spoke 2 would rotate by $-0{,}017\pi$ rad

$$\omega = \frac{0{,}017\pi \text{ rad}}{\frac{1}{30}\text{ sec}} = 0{,}5\pi\,\frac{\text{rad}}{\text{s}} \rightsquigarrow f = \frac{\omega}{2\pi} = \frac{0{,}5\pi}{2\pi} = 0{,}25 \text{ Hz}$$

Thus a TV viewer will see clockwise rotating at a speed of 0,25 revolutions/sec.

d) When within $\frac{1}{30}$ second wheel rotates by integer multiple of $\frac{2\pi}{8}$, i.e. applies

$$\omega = \frac{k\cdot\frac{\pi}{4}}{\frac{1}{30}} = k\cdot\frac{30\pi}{4} \rightsquigarrow f = \frac{\omega}{2\pi} = k\cdot\frac{15}{4} = 3{,}75k\ \frac{\text{revolutions}}{\text{second}} \quad \text{for all integer } k$$

References

- McClellan, Schafer and Yoder, Signal Processing First, ISBN 0-13-065562-7., Prentice Hall, Upper Saddle River, NJ 07458. 2003 Pearson Education, Inc.