20SK: Exercise #4

Communication channels

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Introduction

Communication system objective: *Transmit information from one location to another.*

Medium: *Communication channel*.

Information source:

- Content measured by the *entropy* of source in *bits*.
- Appropriate mathematical model: *random process*.

Two channel types for this lab:

- Binary Symmetric Channel (BSC)
- Additive White Gaussian Noise Channel (AWGN)

Introduction

Channel modelling

Information-carrying signal is subject to a variety of changes:

1. *deterministic*: attenuation, distortion (linear, non-linear),
2. *probabilistic*: additive noise, multipath fading.

Deterministic is only a special case of stochastic $\Rightarrow$ mathematical modelling as stochastic dependence.

Both BSC and AWGN models have stochastic properties.

Discrete Memoryless Channel

Definition

Simplest case of stochastic dependence between input and output: $p(y|x)$, conditional probablity of receiving $y \in \mathcal{Y}$ when transmitting $x \in \mathcal{X}$.

Memoryless channel: Output at time-step i depends only on the input at time-step i.

Definition (Discrete memoryless channel, DMC)

A channel that is completely described by alphabets $\mathcal{X}$ and $\mathcal{Y}$ and the *channel transition probability matrix* $\mathbf{P}$, $\forall x \in \mathcal{X}, y \in \mathcal{Y} : p_{ij} = p(y_j|x_i)$.

Channel capacity

In DMC case

Definition (Mutual information between two random variables)

The "amount of information" obtained about RV X through the other RV, Y:

$$I(X;Y) = \sum_{x \in X} \sum_{y \in Y} p(x,y) \log_2 \frac{p(x,y)}{p(x)p(y)}.$$

Definition (Channel capacity in general)

For channel input X and channel output Y can be computed as

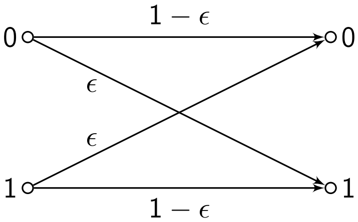
$$C = \max_{\mathbf{p}} I(X;Y).$$

Binary Symmetric Channel

A special case of DMC

Definition (Binary Symmetric Channel, BSC)

A special case of DMC, where $\mathcal{X} = \mathcal{Y} = \{0, 1\}$ and the *crossover probability* $p(y = 0|x = 1) = p(y = 1|x = 0) = \epsilon$ is symmetric.



Binary entropy: $H_\mathrm{b}(x) = -x \log_2 x - (1 - x) \log_2(1 - x)$.
Capacity for BSC: $C = 1 - H_\mathrm{b}(\epsilon)$.

Problem 1

1. Create function `y=bsc(x,eps)` simulating transmission of binary data from vector `x` over BSC with transition probability `eps`.
   1.1 generate a real-valued random vector `err` of the same length as `x`
   1.2 convert it to binary form, where 1 would indicate a crossover
   1.3 invert the value of the original `x` at places where `err` is one
2. What type of noise distribution did we simulate in this example?
3. Does it make sense to test for $\epsilon > 0.5$? Explain.

Problem 1 continued

4. Create function `[eps,ber]=bsc_ber(n)` that will simulate the BER of BSC for packets of length `n` and different values of transition probability.
   Note 1: BER stands for *bit error ratio*, given as a ratio of number of bit errors to the total number of bits transferred.
   Note 2: BER may also denote *bit error rate*, the number of bit errors per unit time.
   4.1 Use `linspace` to define equally spaced crossover probability values ranging from 0.01 to 0.5.
   4.2 Use `bsc` from previous task to compute BSC output for given eps
   4.3 Comparing `x~=y` gives an vector with 1 at places where error occured
   4.4 Summing this vector gives you the number of bit errors
5. Plot the graph $\text{ber} = \text{bsc}(\epsilon)$.

Additive White Gaussian Noise Channel

Review

The channel is bandlimited to $[-W, W]$, the noise is Gaussian and white with power spectral density $N_0/2$ (two-sided). Channel input satisfies power constraint P.

Definition (Capacity of AWGN)

$$C = W \log_2 \left(1 + \frac{P}{N_0 W}\right) = W \log_2 \left(1 + \frac{S}{N}\right),$$

where S/N is the signal-to-noise power ratio (watts, volts, not decibels).

Definition (Q-function)

A probability of error exceeding x is given as

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{u^2}{2}\right) \mathrm{d}u = \frac{1}{2} \operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right)$$

Problem 2

Binary data are transmitted over AWGN with noise power spectral density $N_0/2$ using BPSK with energy E and hard-decision decoding. In this case the channel can be modelled as BSC.

1. Plot the error probability of the channel as a function of

$$\gamma = \frac{E}{N_0}, \qquad \gamma \in [-20\,\mathrm{dB}, 20\,\mathrm{dB}].$$

   1.1 Create vector `gamma_db=-20:0.1:20` and covert it from decibels.
   1.2 The error probability of BPSK with optimal detection is $p = Q(\sqrt{2\gamma})$.
   1.3 Plot it using `semilogx(gamma,p)`.

Problem 2 continued

2. Plot the capacity of the channel as a function of γ.
   2.1 Use the `gamma` from previous task.
   2.2 The capacity of BSC is given by the binary entropy of the channel, which again depends on error probability computed in previous task.
3. What seems to be the limit on error probability? Explain.

Problem 3

1. Plot the capacity of AWGN channel with bandwidth $W = 3000\,\mathrm{Hz}$ as a function of P/N_0 for values of P/N_0 between -20 and $30\,\mathrm{dB}$.
   1.1 adopt the approach we used for `gamma`, i.e. `pn0_db=[-20:0.1:30];` and so on
   1.2 use the standard AWGN capacity formula
2. Now fix the P/N_0 at $25\,\mathrm{dB}$ and plot the capacity as a function of W in the range from 10 to $10000\,\mathrm{Hz}$.
3. Study the limit cases of capacity. Is there any difference between channel behaviour dependent on noise and bandwidth?
4. Open another figure and plot three bandwidth-capacity graphs for $P/N_0 = 25, 40$ and $55\,\mathrm{dB}$. What do you observe?