Signals and codes

Signal and systems

Contents

Examples of signals

Definition: an abstraction of any measurable quantity that is a function of one or more independent variables such as time or space

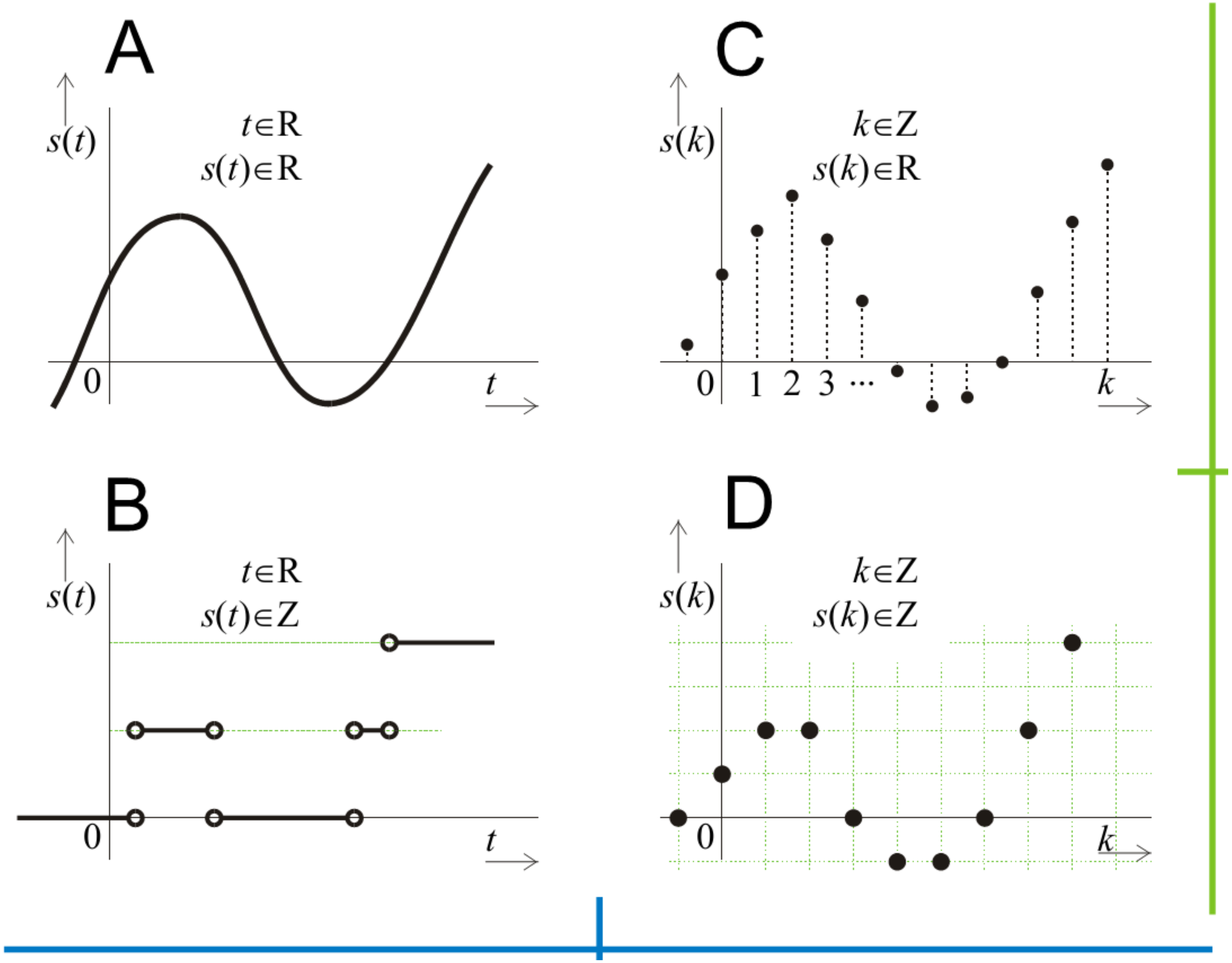
Examples:

- A voltage in a circuit
- A current in a circuit
- Electrocardiograms
- $A\sin(\omega t + \varphi)$
- Speech/music
- Force exerted on a shock absorber
- Image, etc.

Continuous, discrete, odd and even signals
time shift, reversal, scaling transformations

Signal types and transformations

Types of signals



Signals continuous in value

Signals discrete in value, Quantized signals

Signals in continuous time

Signals in discrete time, Sampled signals, sequences

Even & odd symmetry

- One of characteristics of signal is symmetry that may be useful for signal analysis. Even signals are symmetric around vertical axis, and Odd signals are symmetric about origin.

- **Even Signal:**
  A signal is referred to as an even if it is identical to its time-reversed counterparts; $x(t) = x(-t)$. [cosine]

- **Odd Signal:**
  A signal is odd if $x(t) = -x(-t)$.
  An odd signal must be 0 at t=0, in other words, odd signal passes the origin. [sine]

Signal transformations

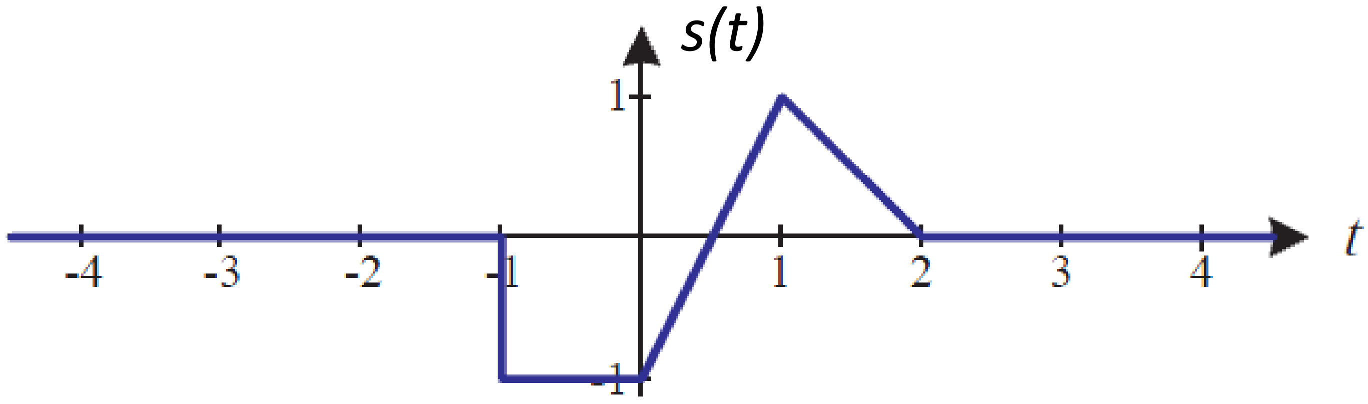
- Time shift: $s(t-t_0)$ and $s[k-k_0]$
  - If $t_0 > 0$ or $k_0 > 0$, signal is shifted to the right
  - If $t_0 < 0$ or $k_0 < 0$, signal is shifted to the left

- Time reversal: $s(-t)$ and $s[-k]$

- Time scaling: $s(at)$ and $s[ak]$
  - If $a > 1$, signal is compressed
  - If $1 > a > 0$, signal is stretched

Signal transformations

- Use the signal *s(t)* to draw:
  s(-t), s(t-1), s(t+2), s(t/2), s(2t), s(2-2t)



Even & odd symmetry

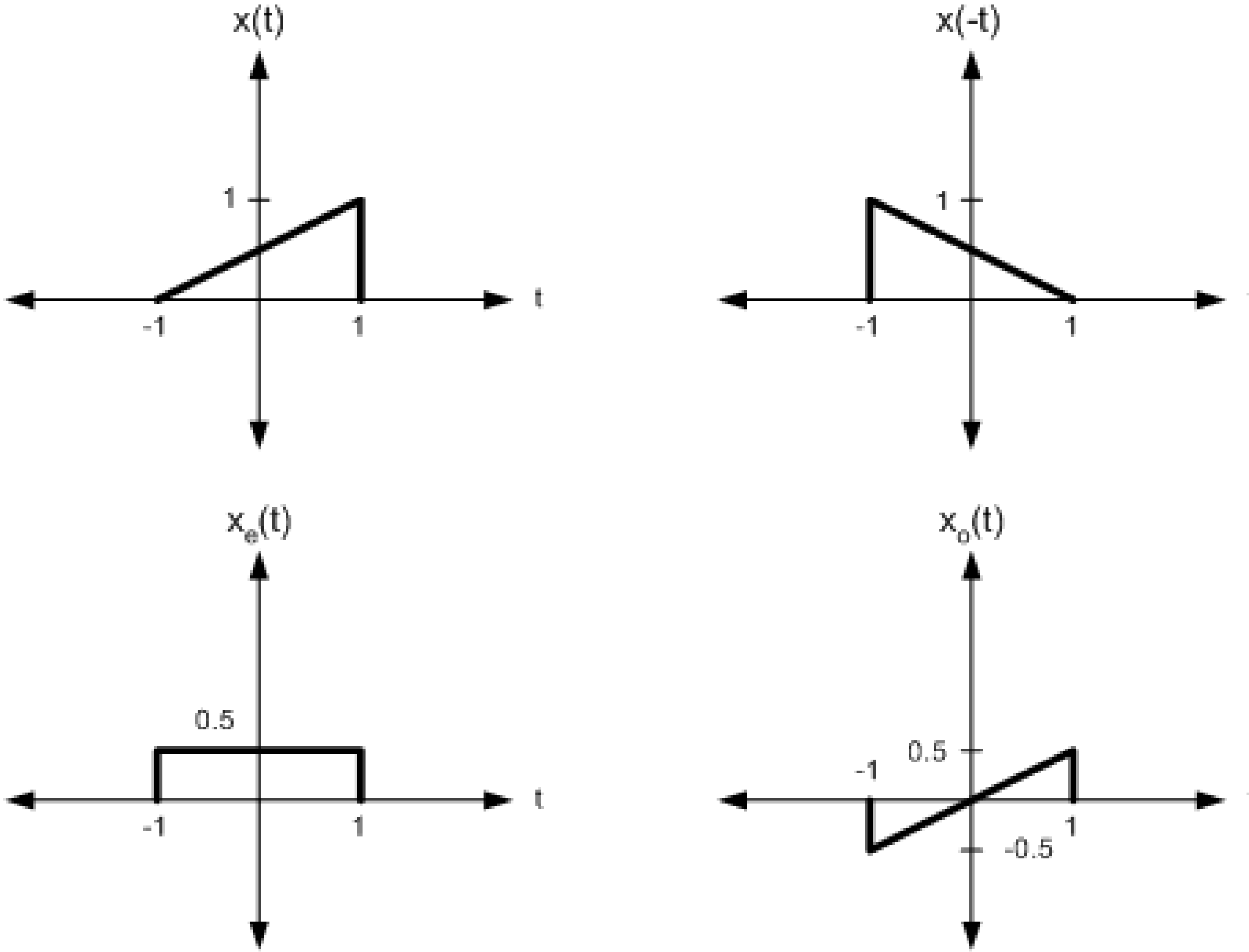
- any signal may be decomposed into a sum of its even part, $x_e(t)$, and its odd part, $x_o(t)$, as follows:

$$\begin{aligned} x_{\mathrm{e}}(t) &= \tfrac{1}{2}\left(x(t) + x(-t)\right) \\ x_{\mathrm{o}}(t) &= \tfrac{1}{2}\left(x(t) - x(-t)\right) \\ x(t) &= x_{\mathrm{e}}(t) + x_{\mathrm{o}}(t) \end{aligned}$$

- It is an important fact because it is <u>relative concept of Fourier series</u>. In Fourier series, a periodic signal can be <u>broken into a sum of sine and cosine signals</u>.
- Notice that sine function is odd signal and cosine function is even signal.

Even & odd symmetry

- Even and odd signals



Average value, effective value, energy, instantaneous power, average power, mutual energy/power, correlation

Signal fundamentals

Basic signal parameters

Notation:

- s(t) … continuous time signals, function of *t*
- s(k) or s[k] … discrete time signals, function of *k*

- **Average value** (DC component)

$$s_{ss} = \mathrm{Av}\left[s(t)\right] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} s(t)\,\mathrm{d}\,t$$

$$s_{ss} = \mathrm{Av}\left[s(k)\right] = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} s(k)$$

Basic signal parameters in DT

- **Energy**

$$E=\sum_{k=-\infty}^{\infty} p(k)=\sum_{k=-\infty}^{\infty}\left|s(k)\right|^{2}=\sum_{k=-\infty}^{\infty} s(k)\cdot s^{*}(k)$$

$$E=\int_{-\infty}^{\infty} p(t)\,\mathrm{d}t=\int_{-\infty}^{\infty}\left|s(t)\right|^{2}\mathrm{d}t=\int_{-\infty}^{\infty} s(t)\cdot s^{*}(t)\,\mathrm{d}t$$

- In signal processing, total **energy** of signal *s*(t) is defined as similar way; (Notice that it is square of absolute value.) where |*s*(t)| denotes the magnitude of *s*(t). It is necessary to get a scalar quantity for complex signal, because magnitude of complex number is defined as $|a+ib|=\sqrt{a^2+b^2}$
- And, it is also squared because of **common convention** to use similar terminology for any signal. Therefore, the energy of a signal is defined as a sum of square of magnitude

Basic signal parameters in DT

- **Average Power**

$$P = \mathrm{Av}\left[p(k)\right] = \mathrm{Av}\left[\left|s(k)\right|^2\right] = \mathrm{Av}\left[s(k)s^*(k)\right]$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} s(k)s^*(k)$$

$$P = \mathrm{Av}\left[p(t)\right] = \mathrm{Av}\left[\left|s(t)\right|^2\right] = \mathrm{Av}\left[s(t)s^*(t)\right]$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} s(t)s^*(t)\,\mathrm{d}t$$

Basic signal parameters in DT

- **Instantaneous Power**

$$p(t) = |s(t)|^2 = s(t) \cdot s^*(t)$$

$$p(k) = |s(k)|^2 = s(k) \cdot s^*(k)$$

- **Effective value**

$$s_{ef} = \sqrt{P}$$

- Electrical definition: the value of an alternating current or voltage equal to the square root of the arithmetic mean of the squares of the instantaneous values taken throughout one complete cycle (of average power)

Energy and Power signals

Energy signal

- has a <u>finite</u> energy, $0 < E < \infty$.
- energy signals have values only in the limited time duration.
- a signal having only one square pulse is energy signal.
- The power of an energy signal is 0, because of dividing finite energy by infinite time (or length).

Power signal

- has a <u>finite</u> power, $0 < P < \infty$.
- **power signal** is not limited in time. It always exists from beginning to end and it never ends.
- We use power (energy per given time) for power signal, because the power of power signal is finite.

Energy and Power signals

- Rules for determining if signal has finite energy of finite average power

- Signals with finite energy have zero average power

$$E_\infty < \infty \Rightarrow P_\infty = 0$$

- Signals with finite duration and amplitude have finite energy

$$x(t) = 0 \text{ for } |t| > c \Rightarrow E_\infty < \infty$$

- Signals with finite average power have infinite energy

$$P_\infty > 0 \Rightarrow E_\infty = \infty$$

Mutual energy

- Energy of signal addition $s(t) = s_1(t) + s_2(t)$

$$E = \int_{-\infty}^{\infty} \left| s_1(t) + s_2(t) \right|^2 \mathrm{d}t = \int_{-\infty}^{\infty} \left(s_1(t) + s_2(t) \right) \cdot \left(s_1^*(t) + s_2^*(t) \right) \mathrm{d}t =$$

$$= \underbrace{\int_{-\infty}^{\infty} s_1(t) s_1^*(t) \mathrm{d}t}_{E_1} + \underbrace{\int_{-\infty}^{\infty} s_1(t) s_2^*(t) \mathrm{d}t}_{E_{12}} + \underbrace{\int_{-\infty}^{\infty} s_1^*(t) s_2(t) \mathrm{d}t}_{E_{21}} + \underbrace{\int_{-\infty}^{\infty} s_2(t) s_2^*(t) \mathrm{d}t}_{E_2} =$$

$$= E_1 + E_2 + \boxed{E_{12} + E_{21}}$$

Mutual energy

- *Energy of signal addition is not equal to sum of their energy*
- Mutual energy: $E_{12} = E_{21}^* = \int_{-\infty}^{\infty} s_1(t) s_2^*(t) \mathrm{d}t$

Mutual energy and mutual power

Mutual energy: $$E_{12} = E_{21}^{*} = \int_{-\infty}^{\infty} s_1(t) s_2^{*}(t)\,\mathrm{d}t$$

Mutual (average) power:

$$P_{12} = P_{21}^{*} = \mathrm{Av}\left[s_1(t) s_2^{*}(t)\right] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} s_1(t) s_2^{*}(t)\,\mathrm{d}t$$

if $s_1(t) = s_2(t)$

$$E_{12} = E_{21} = E$$
$$P_{12} = P_{21} = P$$

- For discrete time signals:

Mutual energy: $$E_{12} = E_{21}^{*} = \sum_{k=-\infty}^{\infty} s_1(k) \cdot s_2^{*}(k)$$

Mutual (average) power: $$P_{12} = P_{21}^{*} = \mathrm{Av}\left[s_1(k) s_2^{*}(k)\right] = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} s_1(k) s_2^{*}(k)$$

Orthogonal signals

- Two signals are orthogonal if $E_{12} = E_{21} = 0$ for energetic signals or $P_{12} = P_{21} = 0$ for power signals

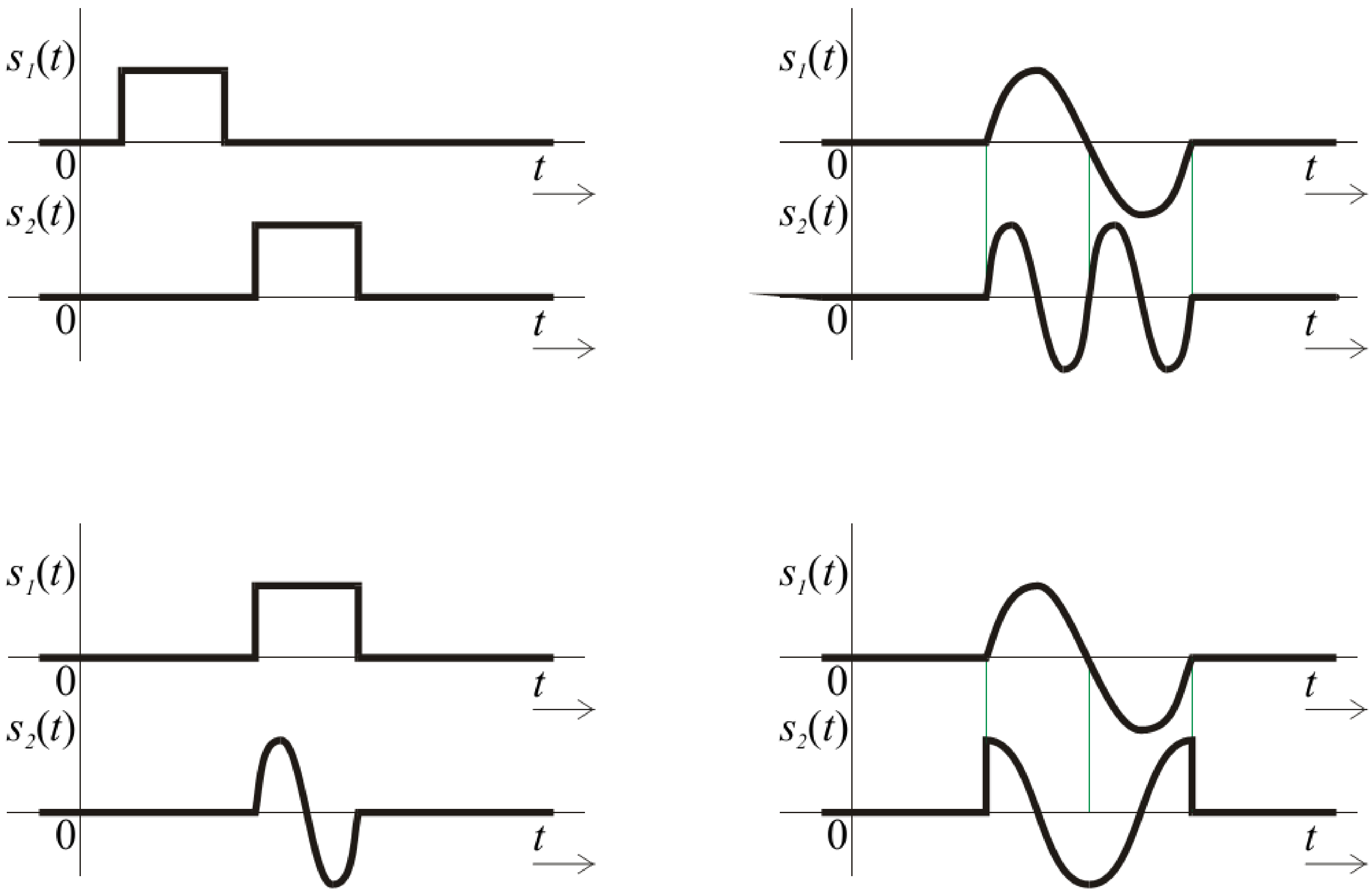
Why?

- Signals that are orthogonal can be separated from each other. They can be therefore used for transmission of different information using one channel (sharing)[1].
- E_{12} and P_{12} can be used as criterion for similarity of signals = used for detection of presence of a signal
- *"I"* denotes interval of orthogonality

$$E_{12} = \int_{(I)} s_1(t) \cdot s_2^*(t)\,\mathrm{d}t = 0$$

1 http://en.wikipedia.org/wiki/Orthogonal_frequency-division_multiplexing#Orthogonality

Orthogonal signals



Correlation function

- correlation function (mutual energy or mutual power between two different shifted signals (at two different points in space or time) = **cross correlation**

$$R_{12}(\tau) = \int_{-\infty}^{\infty} s_1(t+\tau)\, s_2^*(t)\,\mathrm{d}t \qquad R_{12}(0) = E_{12}$$

- correlation function between random variables representing the same quantity measured at two different points then this is often referred to as an **autocorrelation**

$$R(\tau) = \int_{-\infty}^{\infty} s(t+\tau)\, s^*(t)\,\mathrm{d}t \qquad R(0) = E$$

Correlation function

- Cross correlation function (mutual energy or power between two different time or space shifted signals)

$$R_{12}(\tau) = \int_{-\infty}^{\infty} s_1(t+\tau)s_2^*(t)\,\mathrm{d}t \qquad R_{12}(\tau) = \mathrm{Av}\left[s_1(t+\tau)s_2^*(t)\right]$$

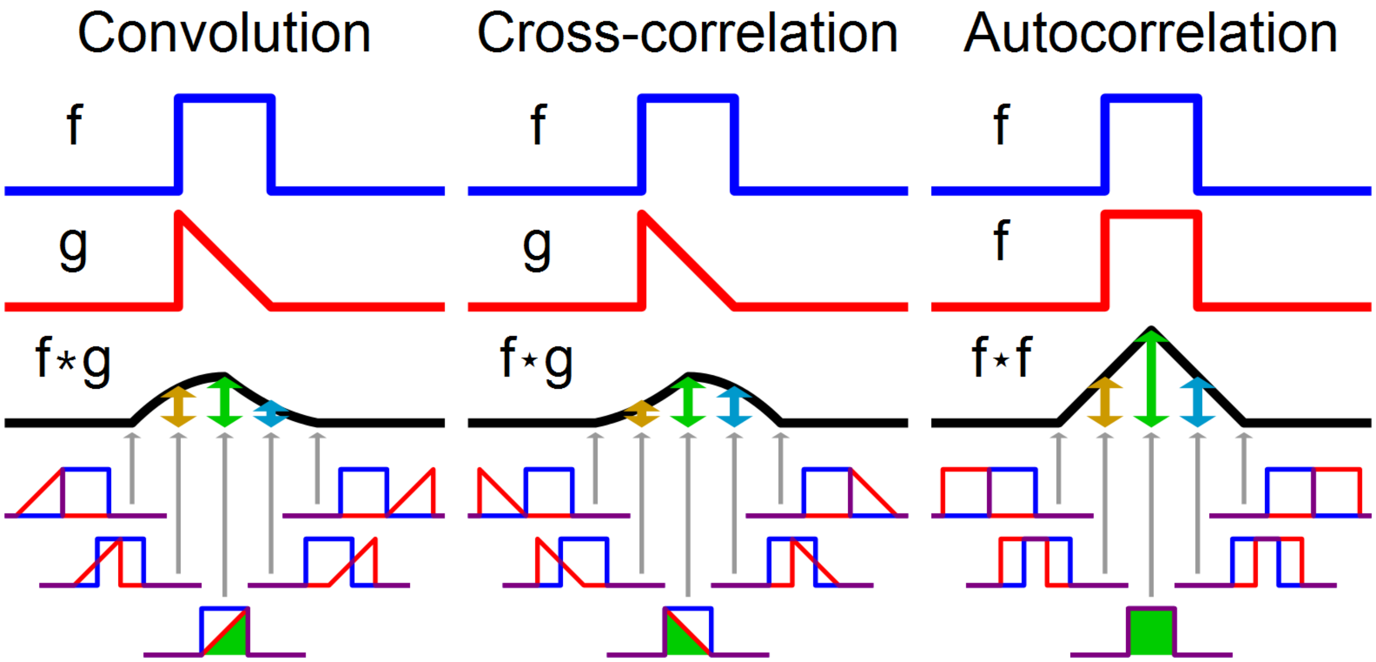
$$R_{12}(m) = \sum_{k=-\infty}^{\infty} s_1(k+m)\cdot s_2^{\;*}(k) \qquad R_{12}(m) = \mathrm{Av}\left[s_1(k+m)s_2^*(k)\right]$$

- Auto correlation function (mutual energy or power between one (same) time or space shifted signal)

$$R(\tau) = \int_{-\infty}^{\infty} s(t+\tau)s^*(t)\,\mathrm{d}t \qquad R(\tau) = \mathrm{Av}\left[s(t+\tau)s^*(t)\right]$$

$$R(m) = \sum_{k=-\infty}^{\infty} s(k+m)\cdot s^*(k) \qquad R(m) = \mathrm{Av}\left[s(k+m)s^*(k)\right]$$

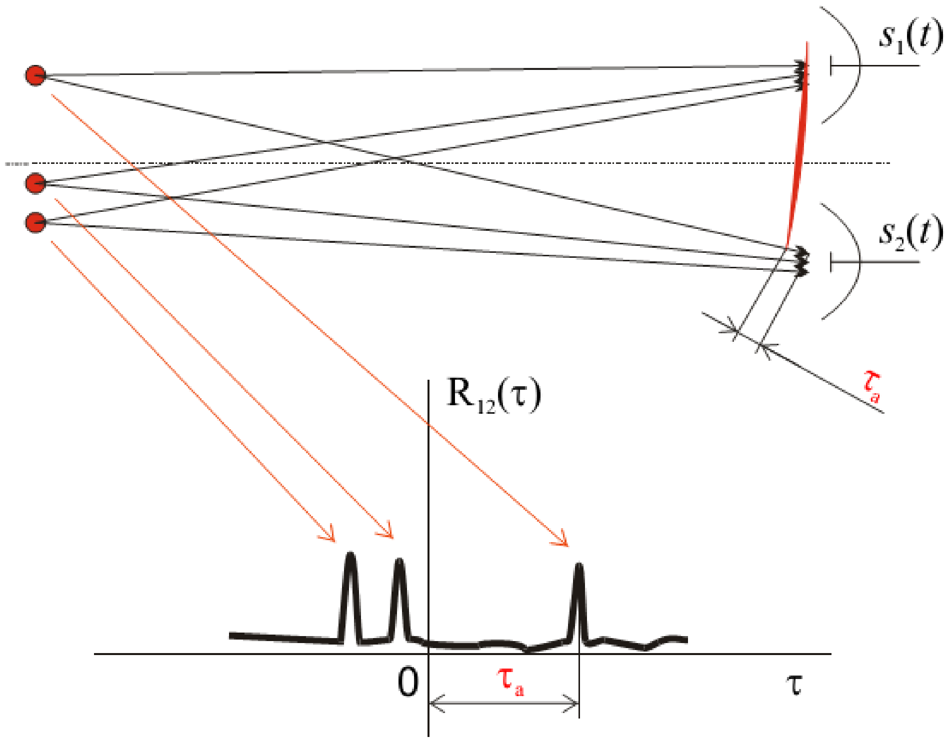
Correlation function



Source: http://en.wikipedia.org/wiki/Correlation_function

Correlation function

- indicator of dependencies as a function of distance in time or space,



- can be used to assess the distance required between sample points for the values to be effectively uncorrelated.

Periodic signal

- Always a **power signal**
- Average value can be computed over a period or integer number of periods. (concerns the Av[] operator)

$$\mathrm{Av}\left[s(t)\right]=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}s(t)\,\mathrm{d}t=\lim_{m\to\infty}\frac{1}{2mT_0+2\nu}\int_{-mT_0-\nu}^{mT_0+\nu}s(t)\,\mathrm{d}t=$$

$$=\underbrace{\lim_{m\to\infty}\frac{1}{2mT_0+2\nu}\int_{-mT_0-\nu}^{-mT_0}s(t)\,\mathrm{d}t}_{\to 0}+\lim_{m\to\infty}\frac{1}{2mT_0+2\nu}\int_{-mT_0}^{mT_0}s(t)\,\mathrm{d}t+\underbrace{\lim_{m\to\infty}\frac{1}{2mT_0+2\nu}\int_{mT_0}^{mT_0+\nu}s(t)\,\mathrm{d}t}_{\to 0}=$$

$$=\lim_{m\to\infty}\frac{1}{2mT_0+2\nu}\int_{-mT_0}^{mT_0}s(t)\,\mathrm{d}t=\lim_{m\to\infty}\frac{1}{2mT_0}\int_{-mT_0}^{mT_0}s(t)\,\mathrm{d}t=\frac{1}{T_0}\int_{(T_0)}s(t)\,\mathrm{d}t$$

$$\mathrm{Av}\left[s(t)\right]=\frac{1}{T_0}\int_{(T_0)}s(t)\,\mathrm{d}t$$

$$\mathrm{Av}\left[s(k)\right]=\frac{1}{N_0}\sum_{k=k_0}^{k_0+N_0-1}s(k)$$

Periodic signals

- Average value, average power and correlations for time continuous and discrete time signals

$$\mathrm{Av}\left[s(t)\right] = \frac{1}{T_0}\int_{(T_0)} s(t)\,\mathrm{d}t$$

$$P = \mathrm{Av}\left[\left|s(t)\right|^2\right] = \frac{1}{T_0}\int_{(T_0)} \left|s(t)\right|^2\,\mathrm{d}t$$

$$\mathrm{R}_{12}(\tau) = \frac{1}{T_0}\int_{(T_0)} s_1(t+\tau)\cdot s_2^{*}(t)\,\mathrm{d}t$$

$$\mathrm{Av}\left[s(k)\right] = \frac{1}{N_0}\sum_{k=k_0}^{k_0+N_0-1} s(k)$$

$$P = \mathrm{Av}\left[\left|s(k)\right|^2\right] = \frac{1}{N_0}\sum_{k=k_0}^{k_0+N_0-1} \left|s(k)\right|^2$$

$$\mathrm{R}_{12}(m) = \frac{1}{N_0}\sum_{k=k_0}^{k_0+N_0-1} s_1(k+m)\cdot s_2^{*}(k)$$

Exercise 1

- Compute energy and power for following signals:

$$x(t) = \begin{cases} 8 & |t| < 5 \\ 0 & \text{otherwise} \end{cases}$$

$$x(t) = \begin{cases} e^{at} & t > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$x[n] = j$$

$$x[n] = e^{j\omega n}$$

$$x[n] = A\cos(\omega n + \phi)$$

- For system with resistor R, where input signal $u(t) = v(t) = V \cdot \cos(\omega_0 t)$ and output signal $y(t) = i(t) = I \cdot \cos(\omega_0 t)$ and system translation function is $i(t) = u(t)/R$, compute following:
  - Average values, instant power, average signal power, instant mutual and average mutual power for/between *u(t)* and *i(t).*
  - Note: power on the resistor is $p=Pv/R$

Trigonometric functions and exponential functions and their unity

Sinusoids and complex exponential

Periodic Signals

- A signal is periodic if there is a positive value of T or N such that $\boldsymbol{s(t)= s(t + T)}$ or $\boldsymbol{s[n] = s[n + N]}$

- The **fundamental period**, T_0, for continuous-time signals is the smallest positive value of T such that $s(t) = s(t +T)$
- The **fundamental period**, N_0, for discrete-time signals is the smallest positive integer of N such that $s[n] = s[n+ N]$
- Signals that are not periodic are said to be **aperiodic**

Exponential and Sinusoidal Signals

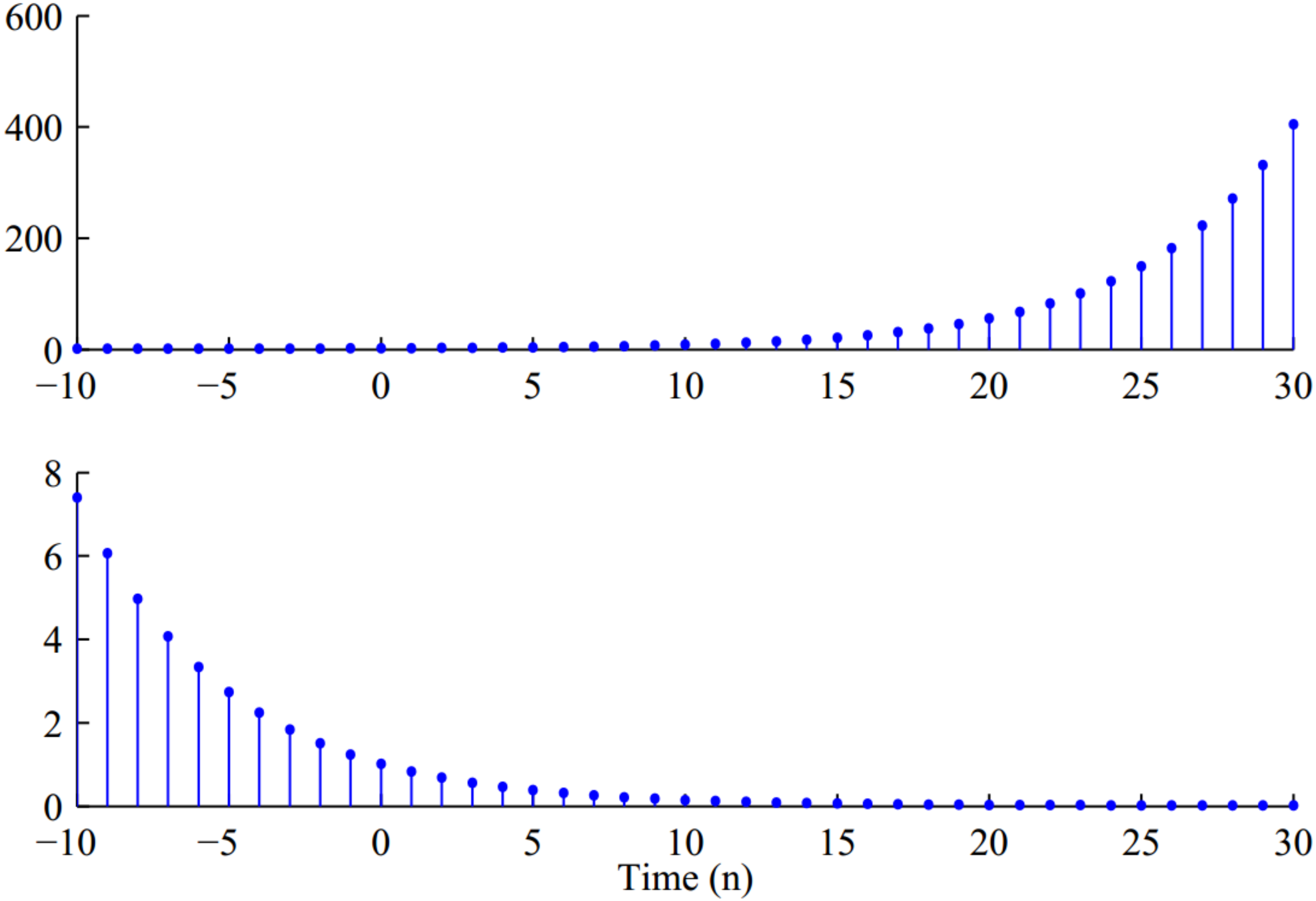
- Exponential signals are:

  $s(t) = A \cdot e^{at}$ or $s[n] = A \cdot e^{an}$

  where A and a are complex numbers.

- Exponential and sinusoidal signals arise naturally in the analysis of linear systems
- Example: simple harmonic motion that you learned in physics
- There are several distinct types of exponential signals
  - A and a real
  - A and a imaginary
  - A and a complex (most general case)

Example $s[n] = A \cdot e^{an}$, $A=1$ and $a=\pm 1/5$



Sinusoidal exponential Signal - Comments

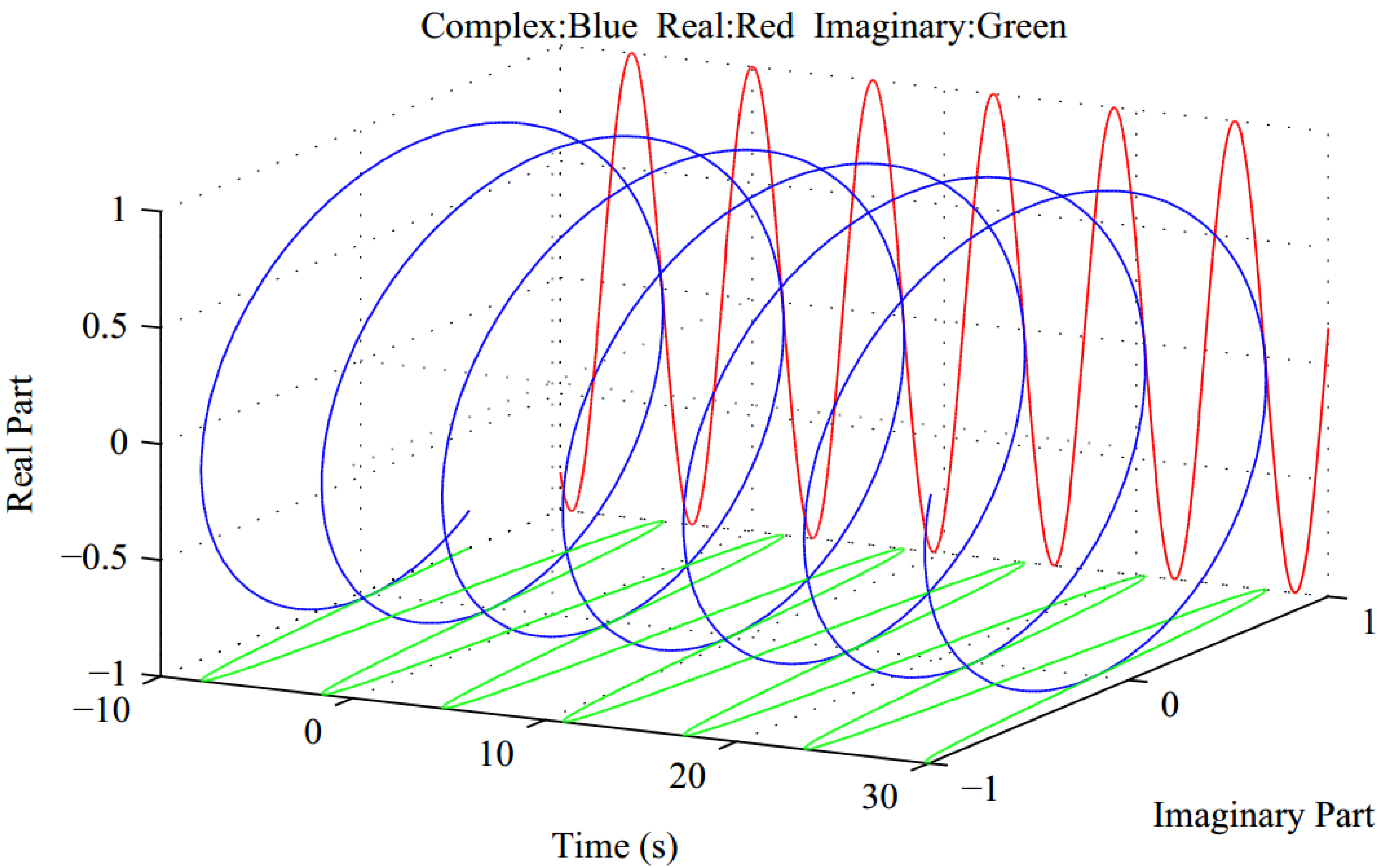
$$s(t) = Ae^{at} = A(e^{a})^{t} = \mathrm{A}\alpha^{t} \qquad s[n] = Ae^{an} = A(e^{a})^{n} = \mathrm{A}\alpha^{n}$$

- When *a* is imaginary, then Euler's equation applies:

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$
$$e^{j\omega n} = \cos(\omega n) + j\sin(\omega n)$$

- Since $\left|e^{j\omega t}\right| = 1$, this looks like a coil in a plot of the complex plane versus time
- $e^{j\omega t}$ is Periodic with fundamental period $T = 2\pi/\omega$
- Real part is sinusoidal: $Re\{Ae^{j\omega t}\} = A\cos(\omega t)$
- Imaginary part is sinusoidal: $\mathrm{Im}\{Ae^{j\omega t}\} = A\sin(\omega t)$
- have infinite energy, but finite (constant) average power, P_{∞}

Example $s(t) = A \cdot e^{at}$, $A=1$ and $a=j$



Sinusoidal Exponential Harmonics

- In order for $e^{j\omega t}$ to be periodic with period T, we require that $e^{j\omega t} = e^{j\omega(t+T)} = e^{j\omega t}e^{j\omega T}$ for all t
- This implies $e^{j\omega T} = 1$ and therefore $\omega T = 2\pi k$ where $k = 0, \pm 1, \pm 2, \ldots$
- There is more than one frequency ω that satisfies the constraint $s(t) = s(t + T)$ where $T = {}^{2\pi k}/_{\omega}$
- The fundamental frequency is given by $k = 1$:

$$\omega_0 = \frac{2\pi}{T_0}$$

- The other frequencies that satisfy this constraint are then integer multiples of ω_0

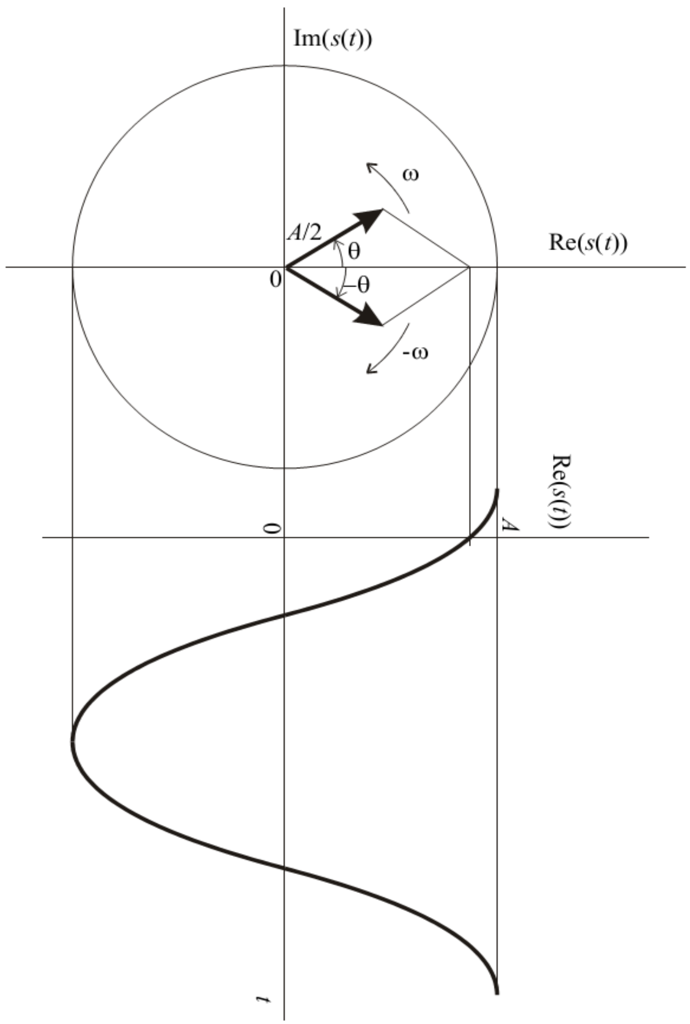
Sinusoidal Exponential Harmonics Continued

- A harmonically related set of complex exponentials is a set of exponentials with fundamental frequencies that are all multiples of a single positive frequency ω_0

$$\emptyset_k(t) = e^{jk\omega_0 t}, \text{where } \; k = 0, \pm 1, \pm 2, \ldots$$

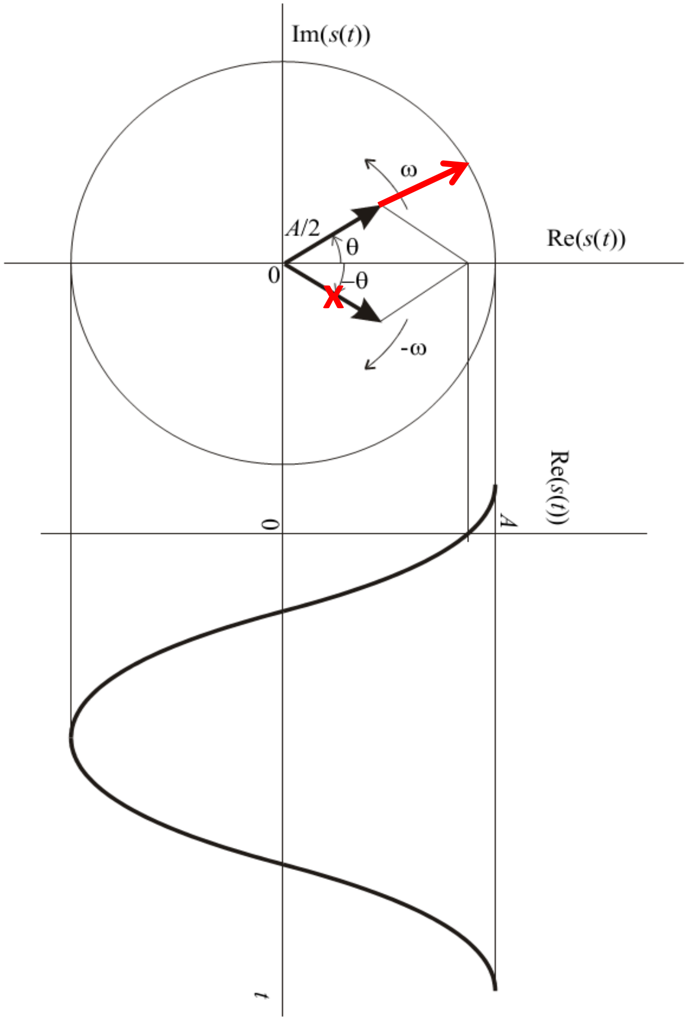
- For $k = 0, \emptyset_k(t)$ is a constant
- For all other values $\emptyset_k(t)$ is periodic with fundamental frequency $|k|\omega_0$
- consistent with how the term **harmonic** is used in music
- Sinusoidal harmonics will play a very important role when we discuss Fourier series and periodic signals

Harmonic signal – real vs. complex



$$A\cos\left(\omega t+\theta\right)=\frac{A}{2}\left(e^{j(\omega t+\theta)}+e^{j(-\omega t-\theta)}\right)$$

Harmonic signal – real vs. complex



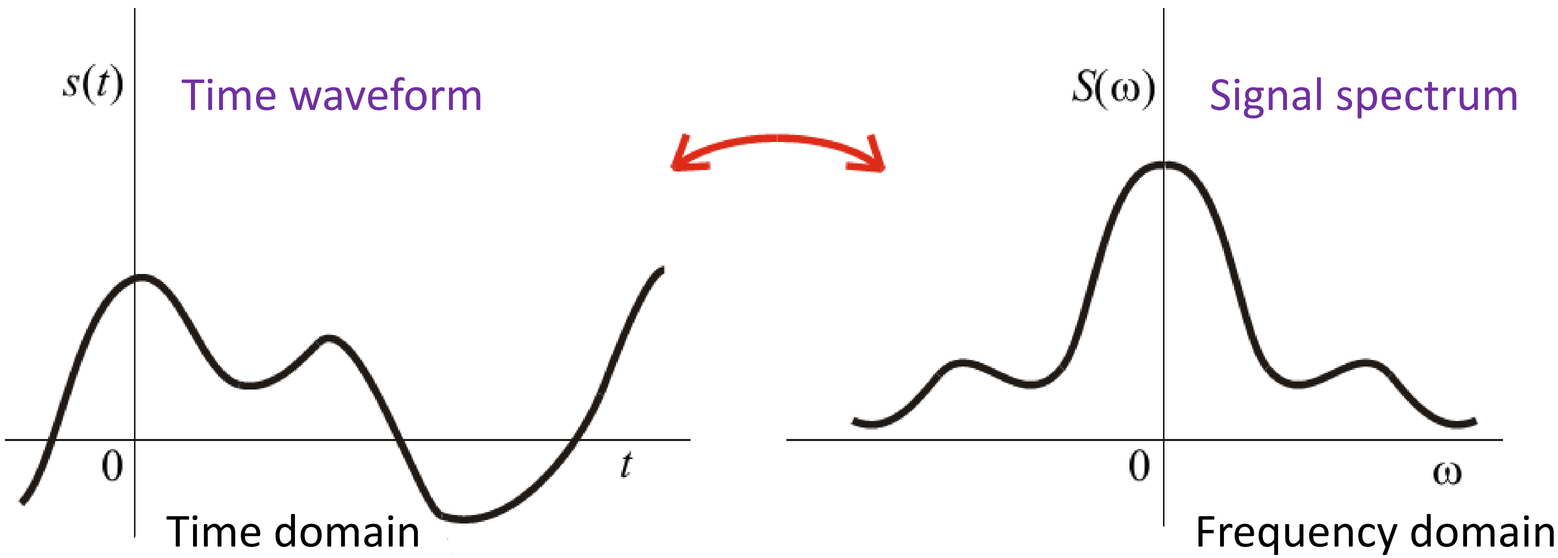
$$A\cos(\omega t+\theta)=\frac{A}{2}\left(e^{j(\omega t+\theta)}+e^{j(-\omega t-\theta)}\right)$$

$$A\cos(\omega t+\theta)=\mathrm{Re}\left(Ae^{j(\omega t+\theta)}\right)$$

Fourier series in trigonometric and complex form, Fourier transform, Parseval's theorem, spectral density, …

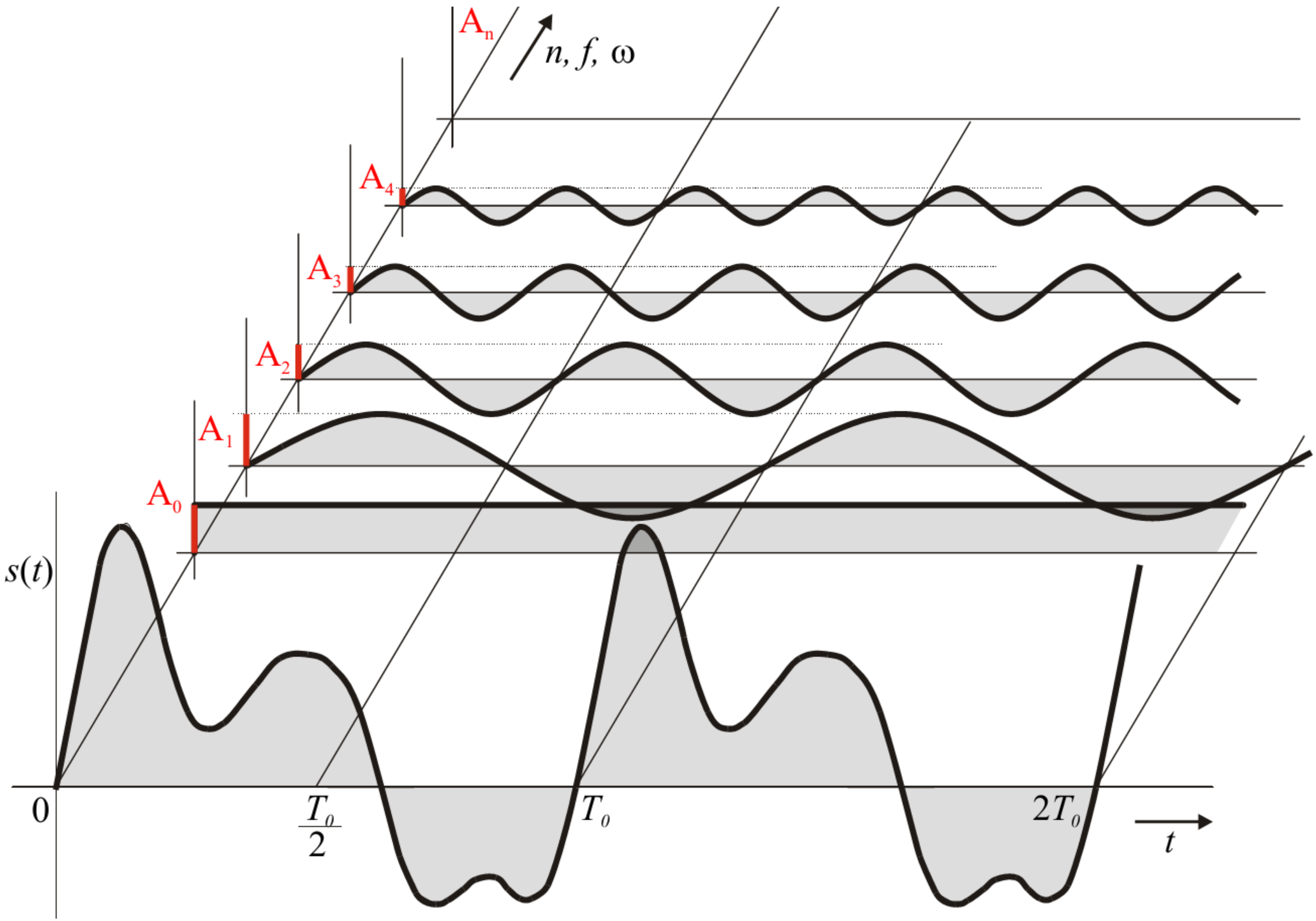
Transformations from time domain into frequency domain and its fundamentals

Spectral representation of signals

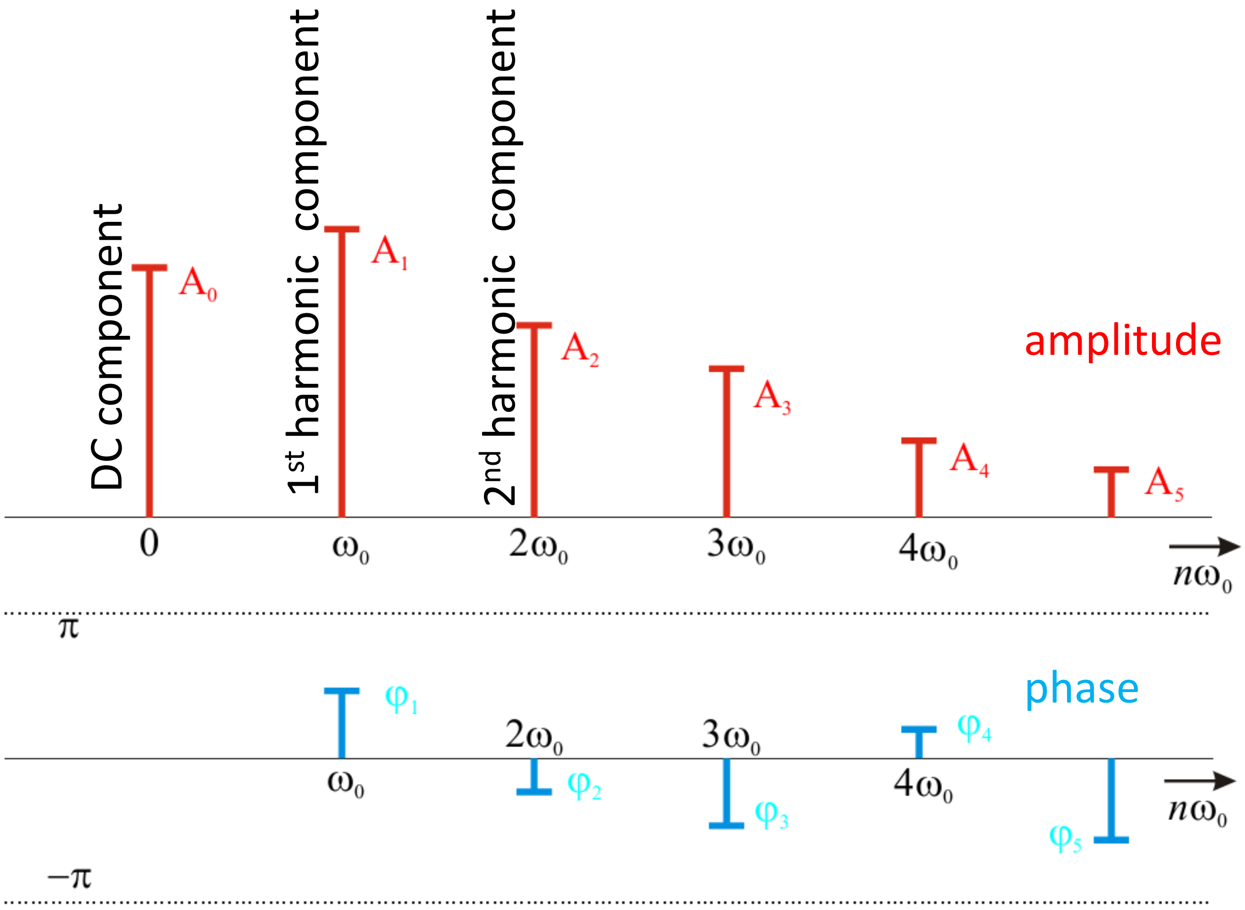


- Performed by
  - Fourier series for periodic signals
  - Fourier transformation for non-periodic signals

Decomposition of periodic signal into Fourier series



Decomposition of periodic signal into Fourier series



$$s(t) = A_0 + A_1 \cdot \cos(\omega_0 t + \varphi_1) + A_2 \cdot \cos(2\omega_0 t + \varphi_2) + A_3 \cdot \cos(3\omega_0 t + \varphi_3) + \ldots$$

Decomposition of periodic signal into Fourier series

- A trigonometric formula:

$$s(t) = A_0 + \sum_{n=1}^{K} A_n \cdot \cos(n\omega_0 t + \varphi_n)$$

- A trigonometric formula (with orthogonal components):

$$s(t) = \frac{a_0}{2} + \sum_{n=1}^{K} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

- Conversion:

$$a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) = A_n \cos(n\omega_0 t + \varphi_n)$$

$$\Rightarrow \qquad a_n = A_n \cos(\varphi_n)$$

$$b_n = -A_n \sin(\varphi_n)$$

$$\frac{a_0}{2} = A_0$$

Decomposition of periodic signal into Fourier series

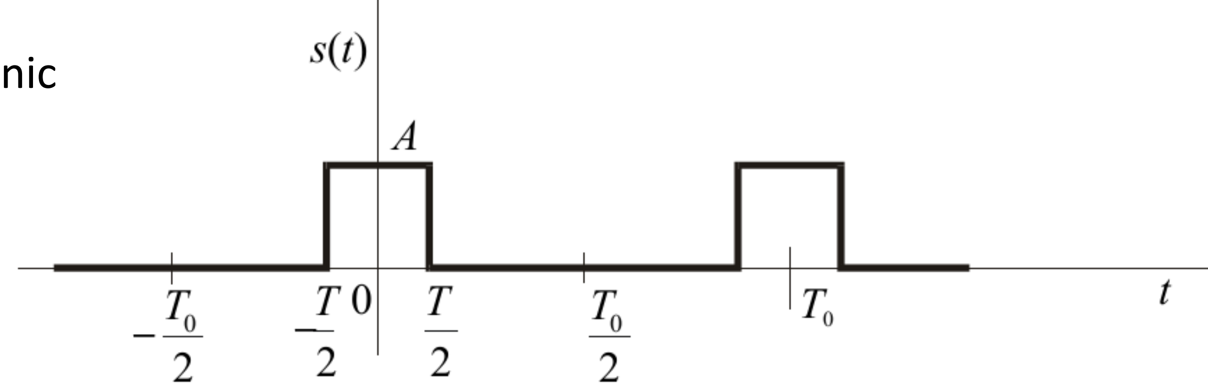
- A trigonometric formula $$s(t)=\frac{a_0}{2}+\sum_{n=1}^{K}a_n\cos(n\omega_0 t)+b_n\sin(n\omega_0 t)$$

- A complex formula: $$\boxed{s(t)=\sum_{n=-K}^{K}c_n e^{jn\omega_0 t}}$$

- Conversion: $$a_n\cos(n\omega_0 t)+b_n\sin(n\omega_0 t)=c_n e^{jn\omega_0 t}+c_{-n}e^{-jn\omega_0 t}$$
$$\frac{a_0}{2}=c_0$$
$$\Rightarrow \quad c_n=c_{-n}^{*}=\frac{a_n-jb_n}{2}$$

- For real signal, the coefficients c_n and c_{-n} are complex conjugate. All information is therefore contained in <u>half of the coefficients</u> (with positive *n*). If the signal is complex, the information is carried by all coefficients c_n.

Example: find coefficients of harmonic Fourier series in complex form:



Solution: define integration interval *I* so it contains just one impulse.

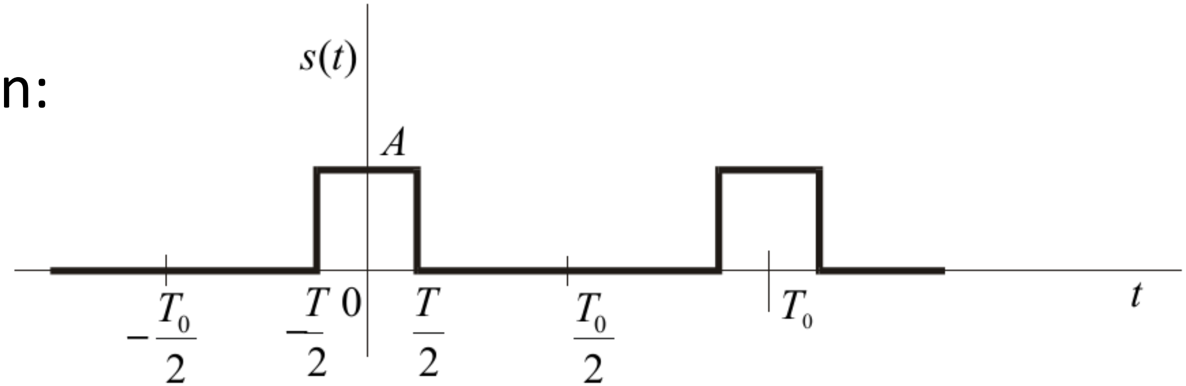
$$c_n = \frac{1}{T_0}\int_{(I)} s(t)\cdot e^{-jn\omega_0 t}\,\mathrm{d}t = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} s(t)\cdot e^{-jn\omega_0 t}\,\mathrm{d}t = \frac{1}{T_0}\int_{-\frac{T}{2}}^{\frac{T}{2}} A\cdot e^{-jn\omega_0 t}\,\mathrm{d}t =$$

$$= \left|\text{for } n\neq 0\right| = \frac{A}{T_0}\left[\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right]_{-\frac{T}{2}}^{\frac{T}{2}} = \frac{A}{T_0 n\omega_0}\cdot\frac{e^{-jn\omega_0\frac{T}{2}} - e^{jn\omega_0\frac{T}{2}}}{-j} =$$

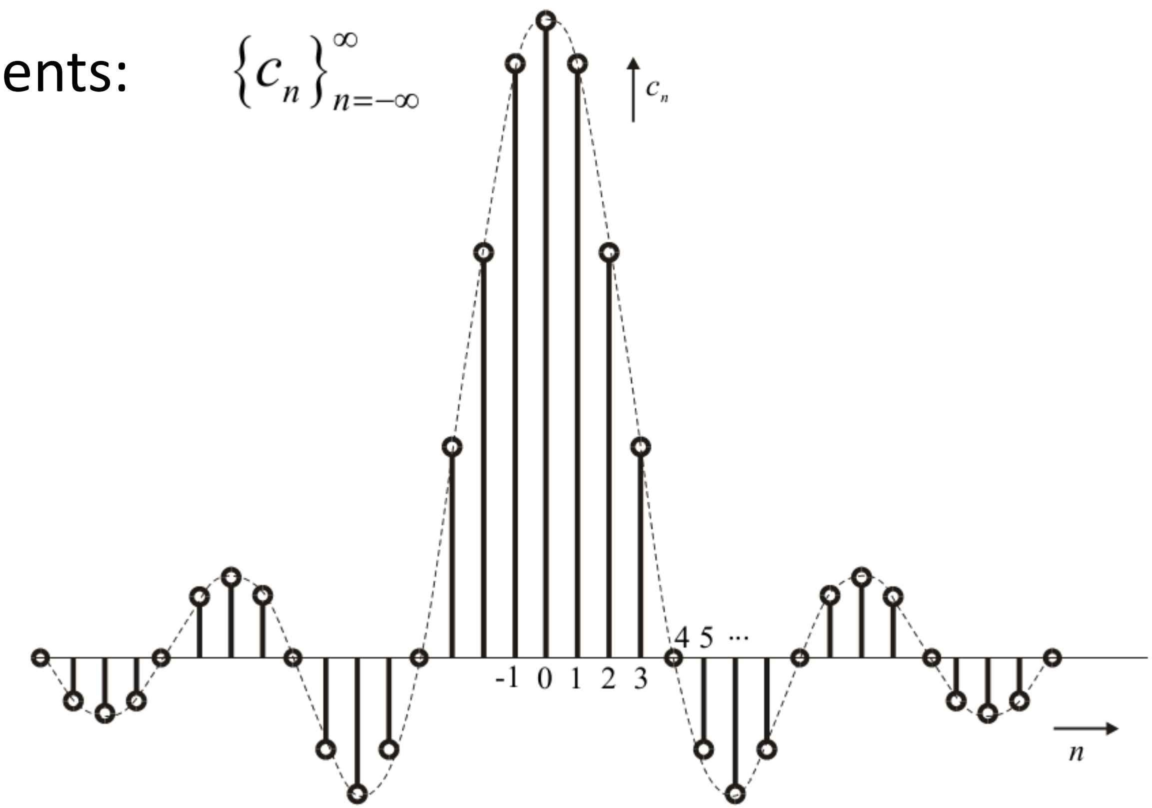
$$= \frac{2A}{T_0 n\omega_0}\cdot\frac{e^{+jn\omega_0\frac{T}{2}} - e^{-jn\omega_0\frac{T}{2}}}{2j} = \frac{2A\frac{T}{2}}{T_0}\cdot\frac{\sin\left(n\omega_0\frac{T}{2}\right)}{n\omega_0\frac{T}{2}} = \frac{AT}{T_0}\cdot\mathrm{Sa}\left(n\omega_0\frac{T}{2}\right)$$

$$c_0 = \frac{1}{T_0}\int_{-\frac{T}{2}}^{\frac{T}{2}} A\cdot e^0\,\mathrm{d}t = \frac{AT}{T_0}$$

Signal in time domain:



Spectrum - FS coefficients: $\{c_n\}_{n=-\infty}^{\infty}$

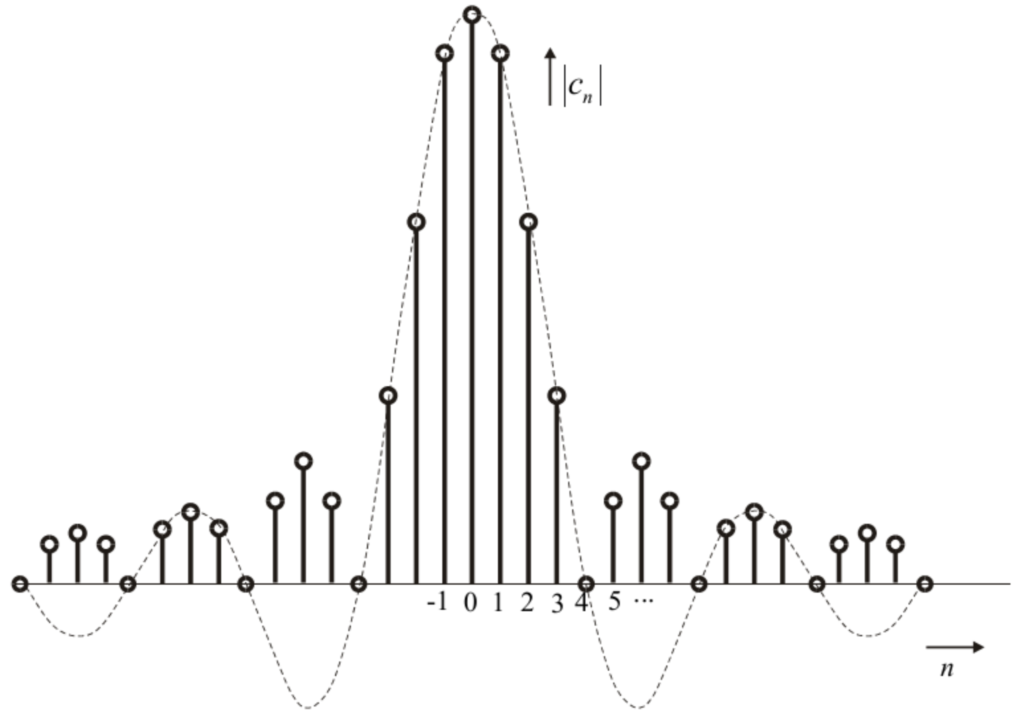


Amplitude spectrum

- coefficients $\left\{\left|c_n\right|\right\}_{n=-\infty}^{\infty}$

Meaning: Amplitude of harmonic components

Always an even sequence for real signals

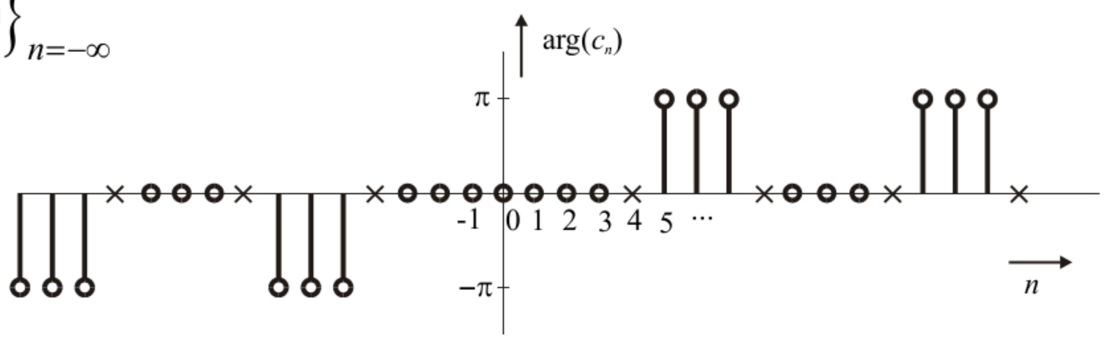


Phase spectrum

- coefficients $\left\{\arg\left(c_n\right)\right\}_{n=-\infty}^{\infty}$

Meaning: phase of harmonic components

Always an odd sequence for real signals

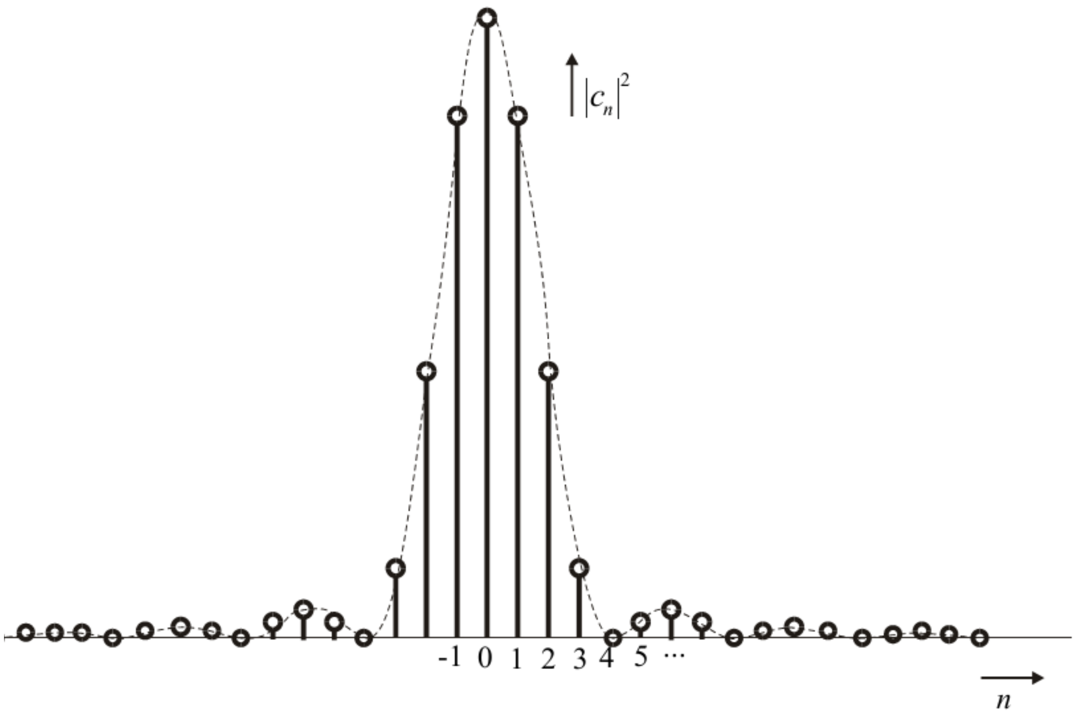


Power spectrum

- coefficients $\left\{\left|c_n\right|^2\right\}_{n=-\infty}^{\infty}$

Meaning: Power in harmonic components

Always an even sequence for real signals



Parseval's theorem

$$\boxed{P = \mathrm{Av}\left[\left|s(t)\right|^2\right] = \sum_{n=-\infty}^{\infty}\left|c_n\right|^2}$$

Power in time domain

Power in spectral (frequency) domain

Generalized Fourier series

$$\boxed{s(t) \cong \sum_{n=n_0}^{n_0+K-1} S_n \cdot \varphi_n(t); \quad t \in I}$$

- Signal approximation by sum of signals in orthogonal system weighted by coefficients S_n
- For given system of functions φ has Fourier sequence minimal error energy from all possible approximations.

$$\boxed{\Delta E_K = \int_I \left| s(t) - \sum_{n=n_0}^{n_0+k-1} S_n \cdot \varphi_n(t) \right|^2 \mathrm{d}t \quad \text{- minimal}}$$

- This condition gives us a formula for calculation of coefficients S_n (minimum delta E_K)

Orthogonal systems

- Orthogonal system: $\left\{\varphi_n(t)\right\}_{n=n_0}^{n_0+K-1}$

$$E_{i,j}=\int_{(I)}\varphi_i(t)\cdot\varphi_j^*(t)\,\mathrm{d}t=\begin{cases}E_i; & i=j\\ 0; & i\neq j\end{cases}\qquad P_{i,j}=\mathrm{Av}\left[\varphi_i(t)\cdot\varphi_j^*(t)\right]=\begin{cases}P_i; & i=j\\ 0; & i\neq j\end{cases}$$

- Orthonormal system:

$$E_{i,j}=\int_{(I)}\varphi_i(t)\cdot\varphi_j^*(t)\,\mathrm{d}t=\begin{cases}1; & i=j\\ 0; & i\neq j\end{cases}\qquad P_{i,j}=\mathrm{Av}\left[\varphi_i(t)\cdot\varphi_j^*(t)\right]=\begin{cases}1; & i=j\\ 0; & i\neq j\end{cases}$$

- Complete orthogonal system:
  – No new signal with non zero energy, orthogonal to all other signals in the system, can be added system.

Fourier series and Fourier transform

- Nonperiodic signal
  = limit case of a finite signal

$$s_T(t) = \begin{cases} s(t) & \textit{for} \quad |t| \le T \\ 0 & \textit{for} \quad |t| > T \end{cases}$$

- Finite signal $s_T(t)$ can be expressed by Fourier series with orthonormal system

$$\varphi_n(t) = \begin{cases} \frac{1}{2T} e^{jn\omega_0 t} & \textit{for} \quad |t| \le T \\ 0 & \textit{for} \quad |t| > T \end{cases} \qquad \omega_0 = \frac{2\pi}{2T}, \ E_n = \frac{1}{2T}$$

$$s_T(t) = \begin{cases} \frac{1}{2T} \sum_{n=-\infty}^{\infty} S_n e^{jn\omega_0 t} = \frac{\omega_0}{2\pi} \sum_{n=-\infty}^{\infty} S_n e^{jn\omega_0 t} & \textit{for} \quad |t| \le T \\ 0 & \textit{for} \quad |t| > T \end{cases}$$

$$S_n = \frac{1}{E_n} \int_{-T}^{T} s_T(t) \frac{1}{2T} e^{-jn\omega_0 t} \, \mathrm{d}t = \int_{-T}^{T} s_T(t) e^{-jn\omega_0 t} \, \mathrm{d}t = \int_{-T}^{T} s(t) e^{-jn\omega_0 t} \, \mathrm{d}t$$

Fourier series and Fourier transform

- Limit transition to nonfinite signal $s(t) = \lim_{T \to \infty} s_T(t)$
- If $T \to \infty$ and $\omega_0 \to 0$, *product of* $n\omega_0$ for n big enough will be nonzero number
- In limit case, for $T \to \infty$ evolve discrete product $n\omega_0$ into continuous quantity ω and discrete spectrum S_n evolves into continuous function $S(\omega)$

$$\lim_{T \to \infty} S_n = \lim_{T \to \infty} \int_{-T}^{T} s(t) e^{-jn\omega_0 t} \,\mathrm{d}t = \int_{-\infty}^{\infty} s(t) e^{-j\omega t} \,\mathrm{d}t$$

$$\boxed{S(\omega) = \int_{-\infty}^{\infty} s(t) e^{-j\omega t} \,\mathrm{d}t}$$

$$\lim_{T \to \infty} s_T(t) = \lim_{\omega_0 \to 0} \frac{\omega_0}{2\pi} \sum_{n=-\infty}^{\infty} S_n e^{jn\omega_0 t} = \lim_{\omega_0 \to 0} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} S_n e^{jn\omega_0 t} \omega_0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{j\omega t} \,\mathrm{d}\omega$$

$$\boxed{s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{j\omega t} \,\mathrm{d}\omega}$$

Fourier transform

- Fourier transform:

$$S(\omega) = \mathrm{F}\left[s(t)\right] = \int_{-\infty}^{\infty} s(t)\, e^{-j\omega t}\, \mathrm{d}t$$

- Inverse Fourier transform:

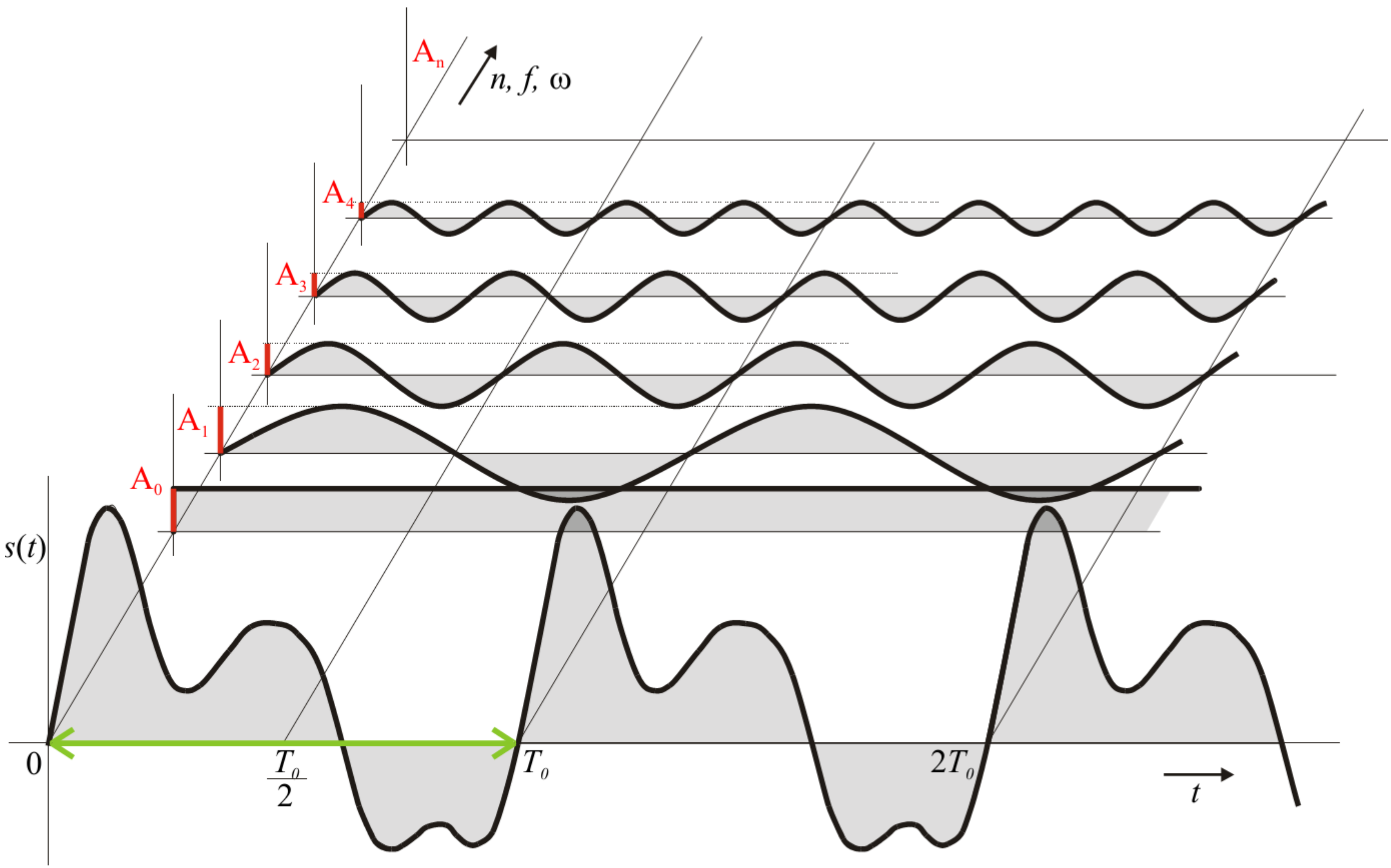
$$s(t) = \mathrm{F}^{-1}\left[S(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, e^{j\omega t}\, \mathrm{d}\omega$$

$S(\omega)$ – Spectrum, for $s(t) \in \mathrm{R}:\ S(-\omega) = S^{*}(\omega)$

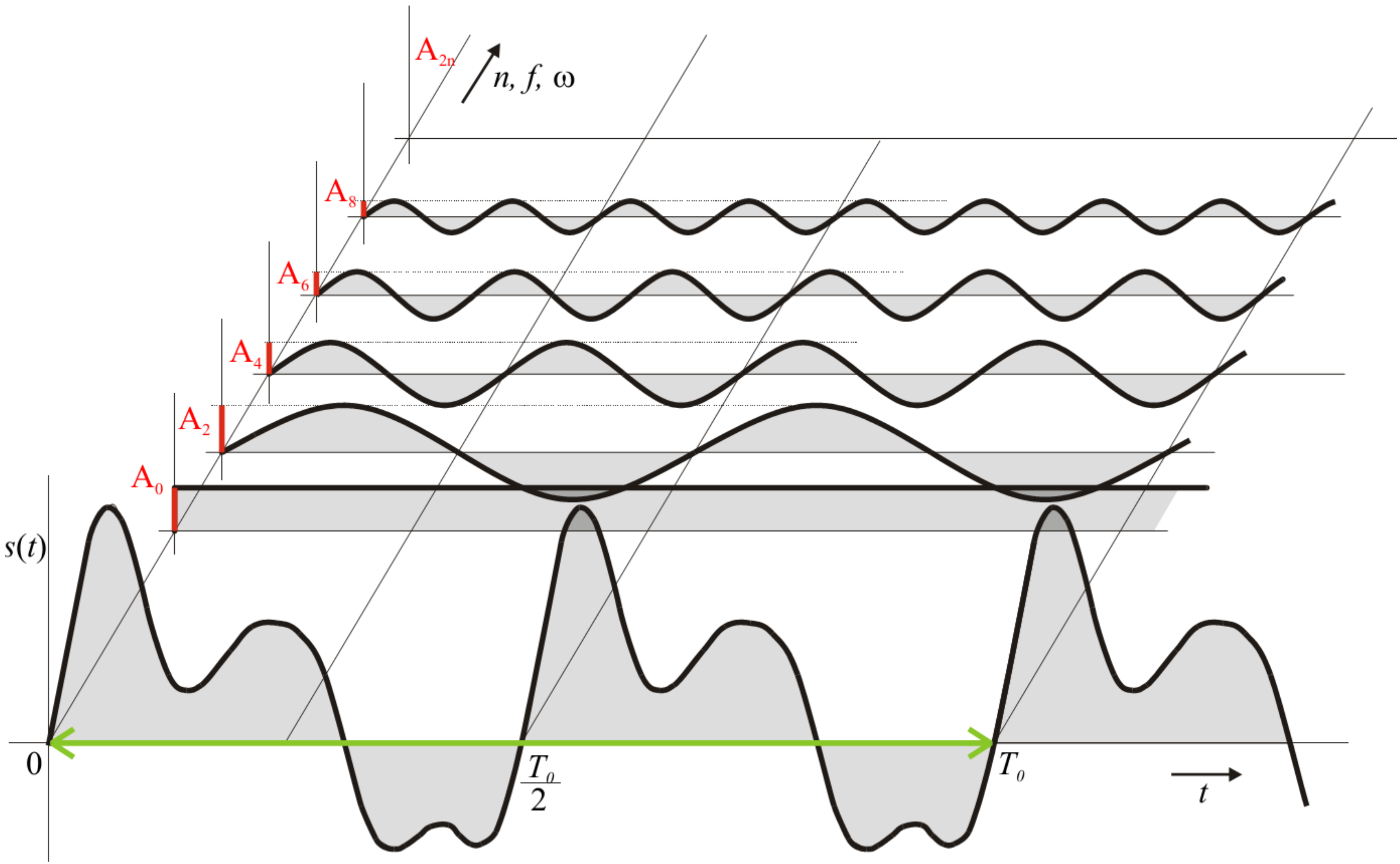
$|S(\omega)|$ – Amplitude spectrum, for $s(t) \in \mathrm{R}:\ |S(-\omega)| = |S(\omega)|$

$\arg(S(\omega))$ – Phase spectrum, for $s(t) \in \mathrm{R}:\ \arg(S(-\omega)) = -\arg(S(\omega))$

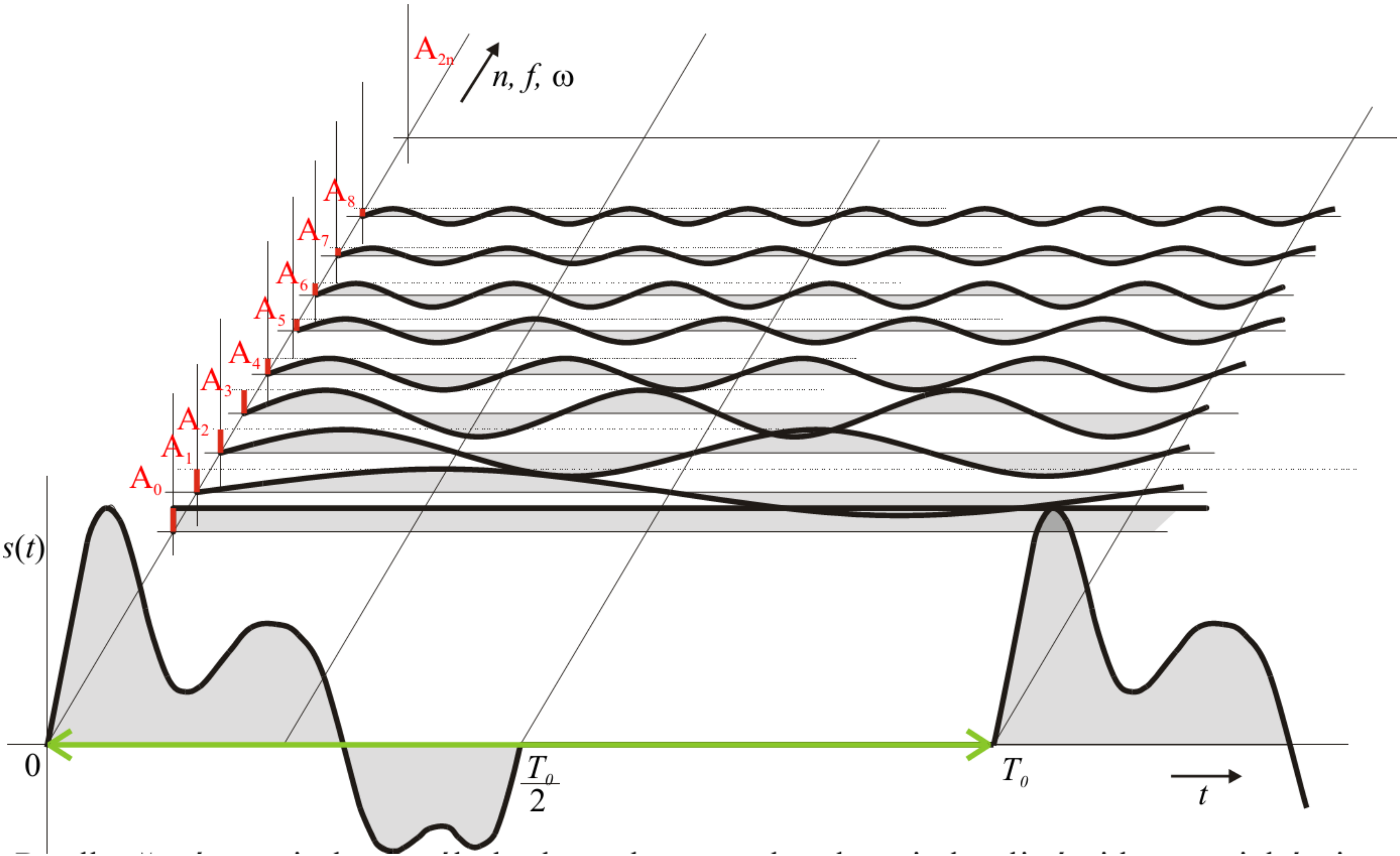
Fourier series and Fourier transform



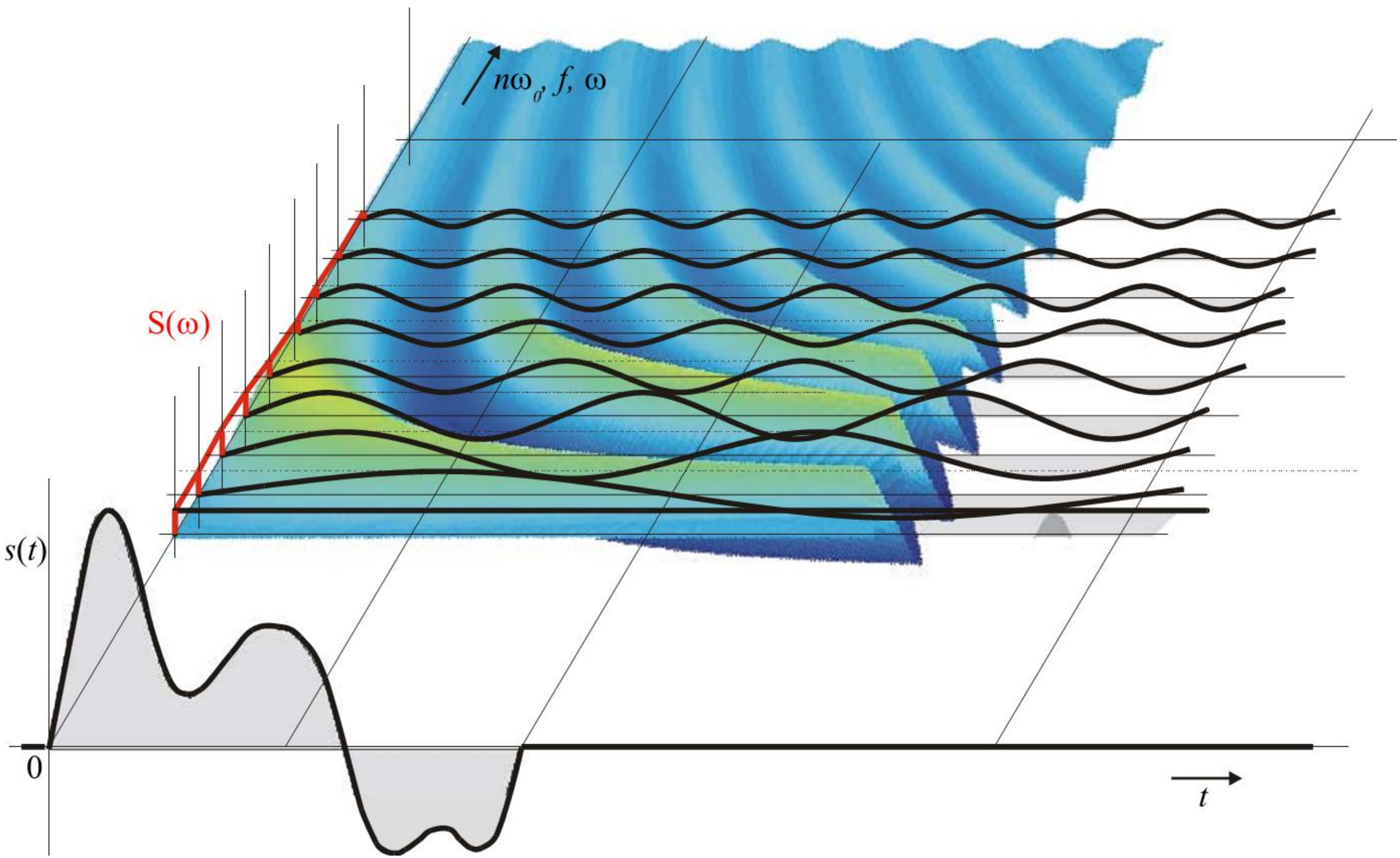
Fourier series and Fourier transform



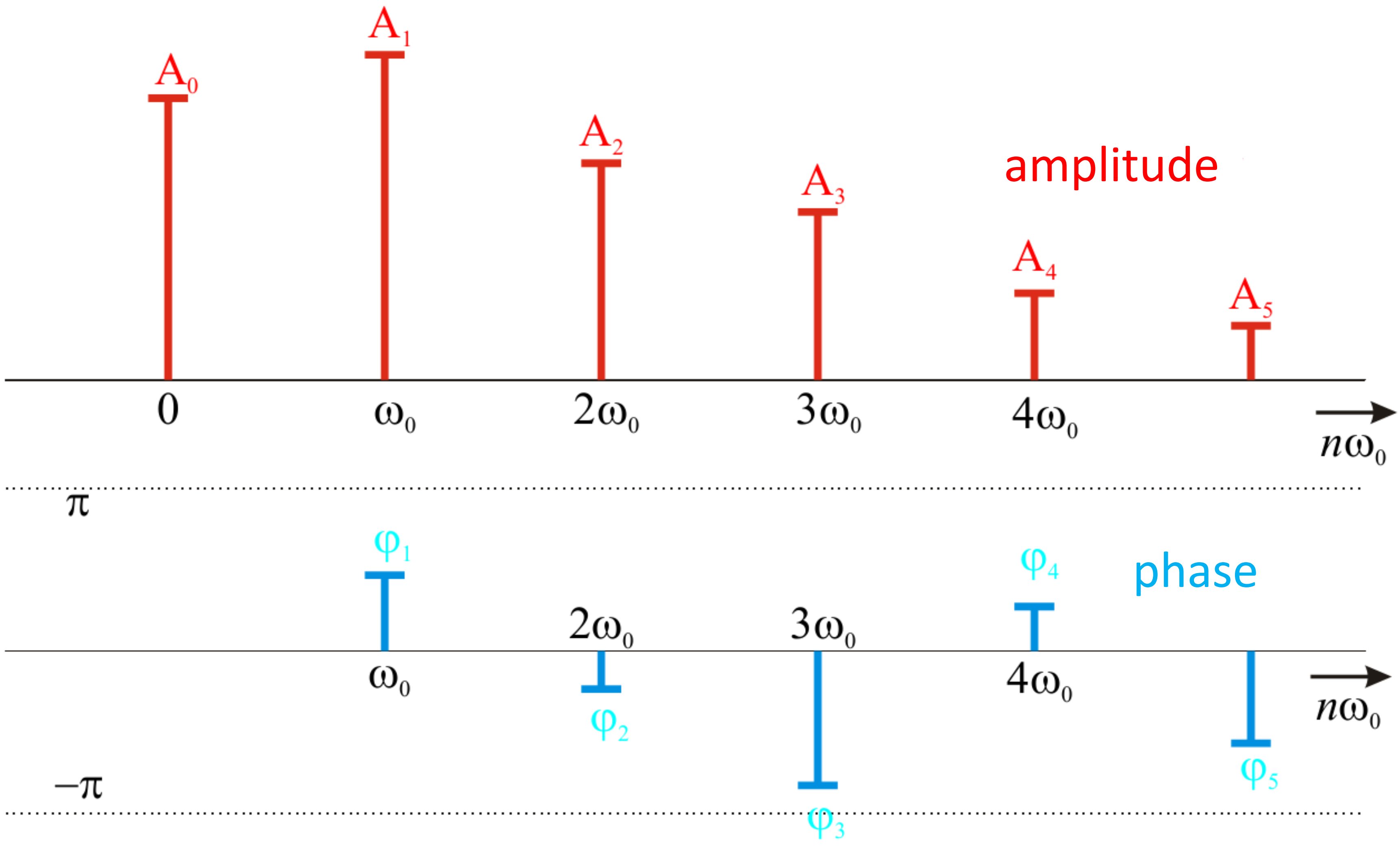
Fourier series and Fourier transform



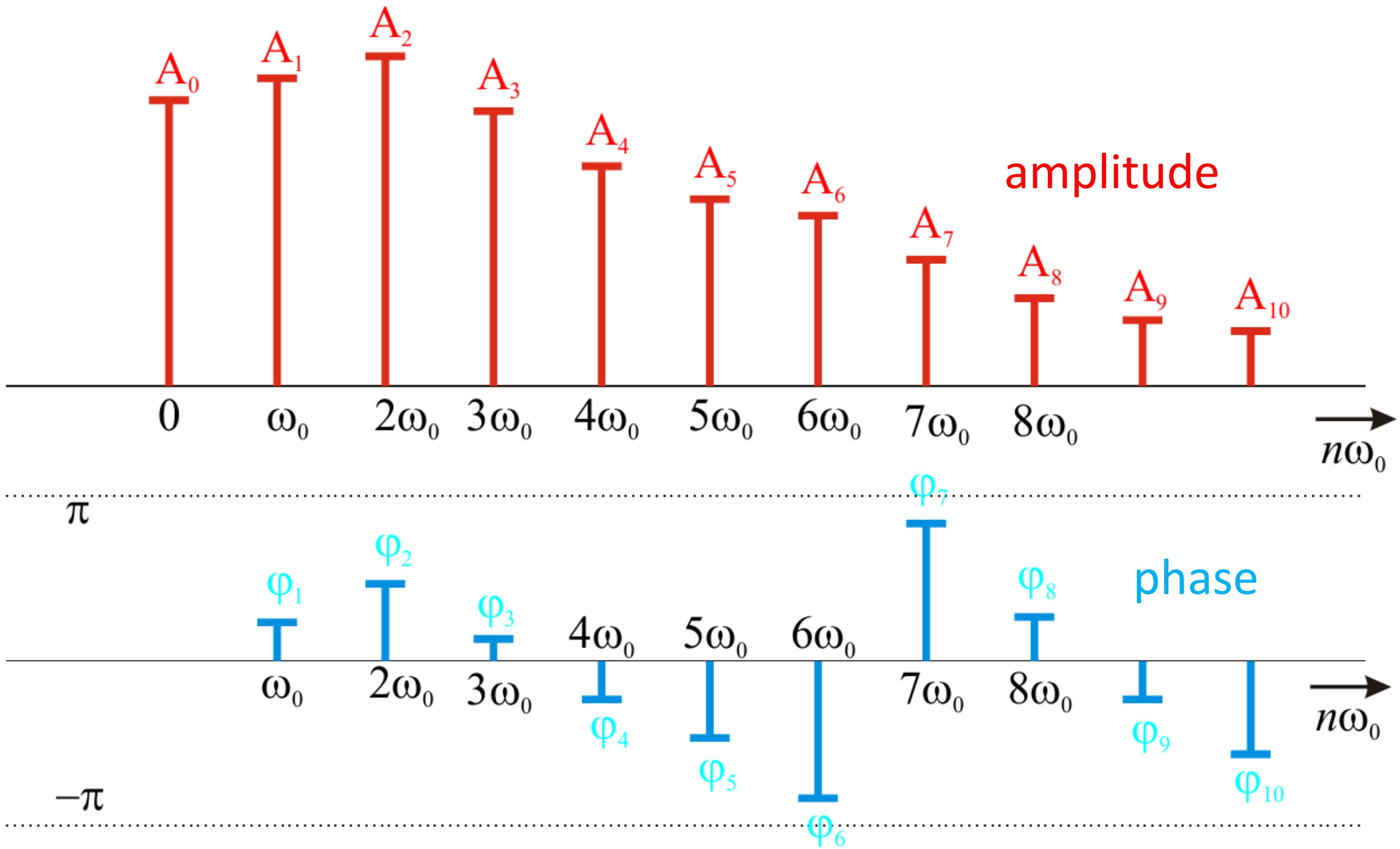
Fourier series and Fourier transform



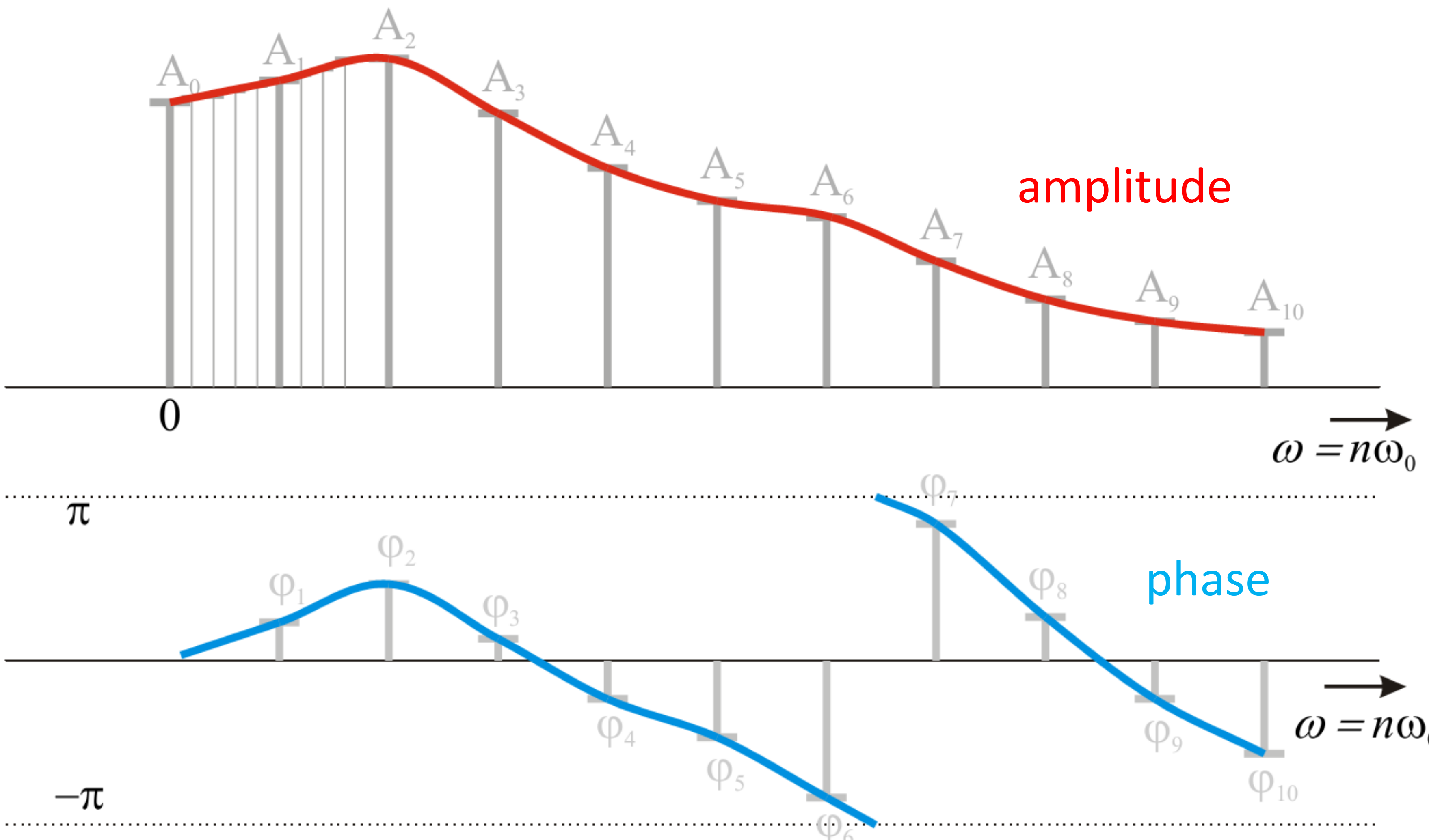
Fourier series coefficients



Fourier series coefficients – period extension



From Fourier series to continuous spectrum



Power spectral density *C(ω)*

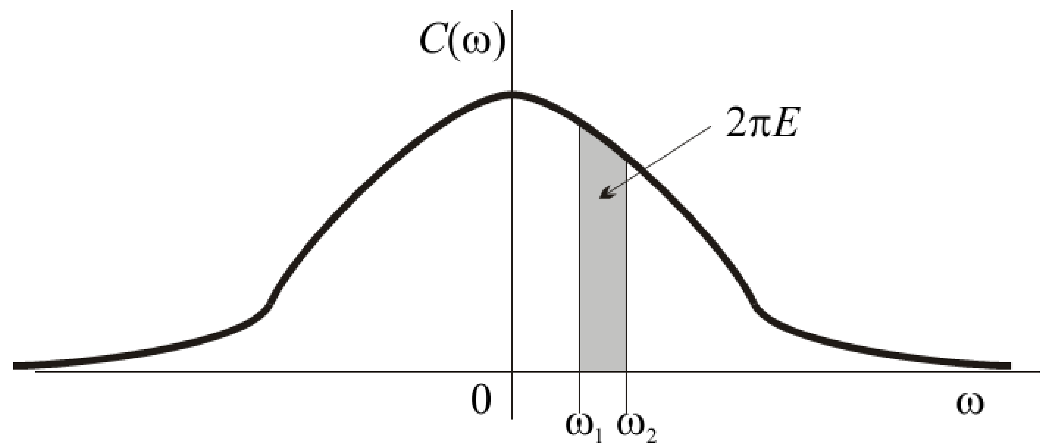
- Energy in whole spectrum (Parseval's theorem)

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\left|S(\omega)\right|^2}_{C(\omega)} d\omega$$

- Energy in part of the spectrum in frequency interval ω_1, ω_2

$$E_{\omega_1,\omega_2} = \frac{1}{2\pi} \int_{\omega_1}^{\omega_2} \underbrace{\left|S(\omega)\right|^2}_{C(\omega)} d\omega$$

$$\boxed{C(\omega) = \left|S(\omega)\right|^2}$$



- Characteristics: positive, even for real signals

Power spectral density

- Energy spectral density of time interval of a signal $s_T(t)$ divided by length of the interval

$$s_T(t) = \begin{cases} s(t) & \mathit{for} \quad |t| \le T \\ 0 & \mathit{for} \quad |t| > T \end{cases}$$

$$S_T(\omega) = \mathrm{F}\left[s_T(t)\right] = \int_{-T}^{T} s(t) \mathrm{e}^{-j\omega t}\, dt$$

$$\boxed{C(\omega) = \lim_{T\to\infty} \frac{1}{2T} \left|S_T(\omega)\right|^2 = \lim_{T\to\infty} \frac{1}{2T} \left| \int_{-T}^{T} s(t) \mathrm{e}^{-j\omega t}\, dt \right|^2}$$

$$P = \frac{1}{2\pi} \int_{-\infty}^{\infty} C(\omega)\, d\omega$$

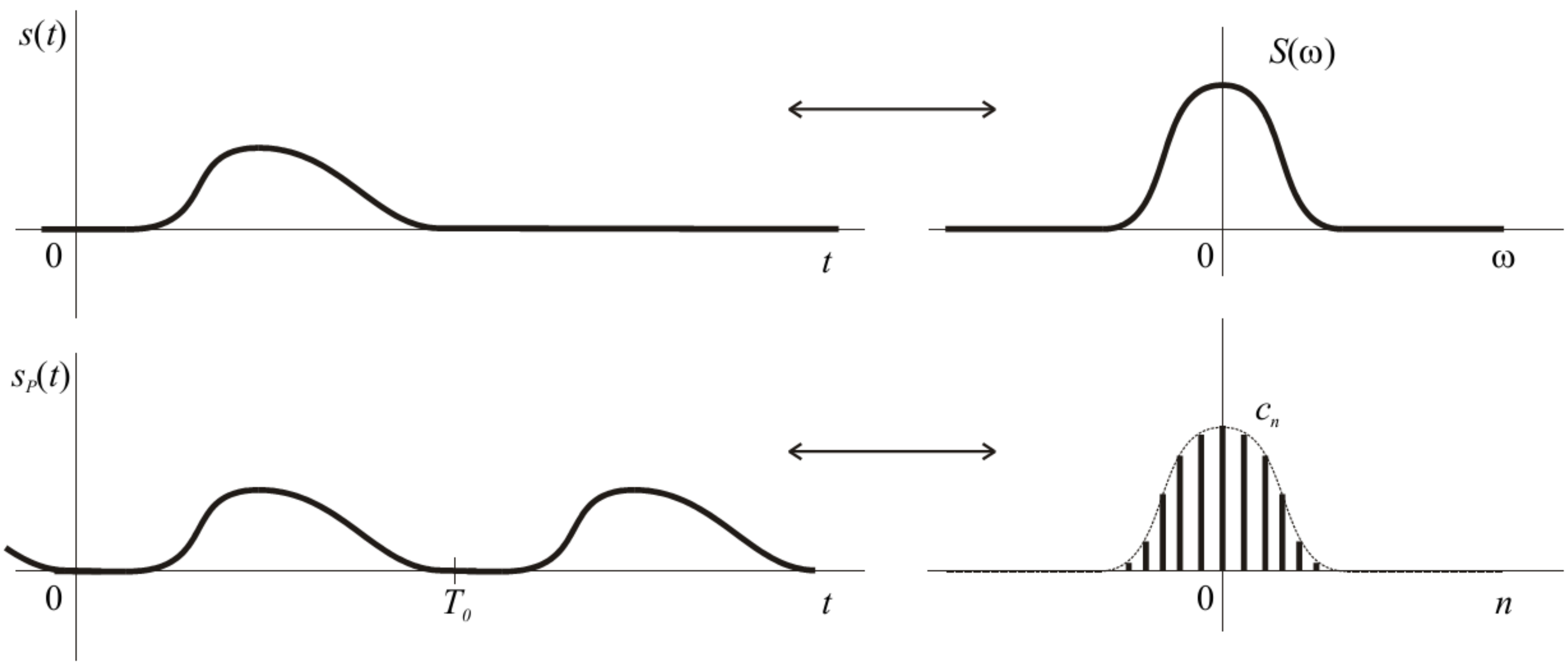
Fourier series and Fourier transform

$$s_P(t) = \sum_{m=-\infty}^{\infty} s(t - mT_0)$$

Periodic signal = Fourier series

$$s(t) = \begin{cases} s_P(t) & ;\ \text{for } t \in \langle 0, T_0 \rangle \\ 0 & ;\ \text{elsewhere} \end{cases}$$

Non-periodic signal = Fourier transform



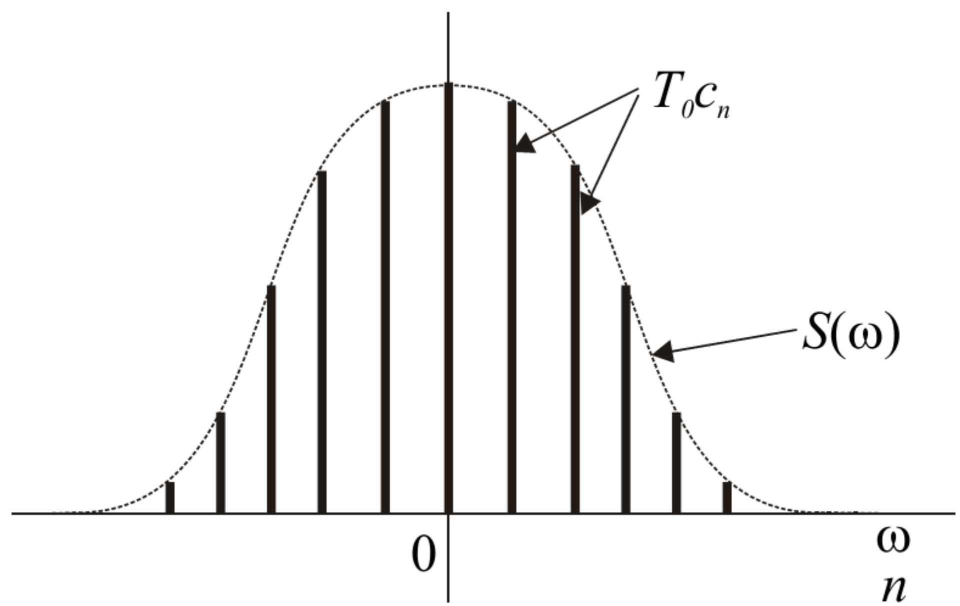
Fourier series and Fourier transform

$$s_P(t) = \sum_{m=-\infty}^{\infty} s(t - mT_0)$$

Periodic signal = Fourier series

$$c_n = \frac{1}{T_0} \int_{(T_0)} s_P(t) \mathrm{e}^{-jn\omega_0 t}\, dt = \frac{1}{T_0} \int_{t_p}^{t_p+T_0} s(t) \mathrm{e}^{-jn\omega_0 t}\, dt = \frac{1}{T_0} \underbrace{\int_{-\infty}^{\infty} s(t) \mathrm{e}^{-jn\omega_0 t}\, dt}_{S(n\omega_0)}$$

$$\boxed{c_n = \frac{1}{T_0} S(n\omega_0)}$$

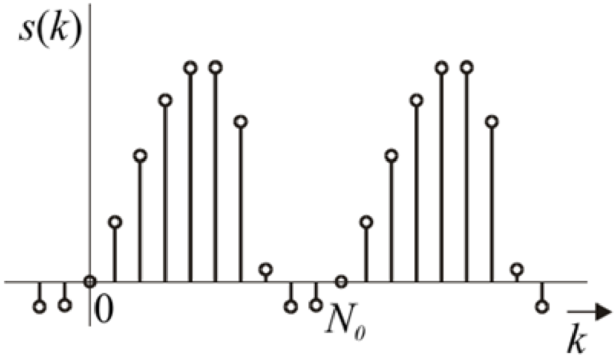


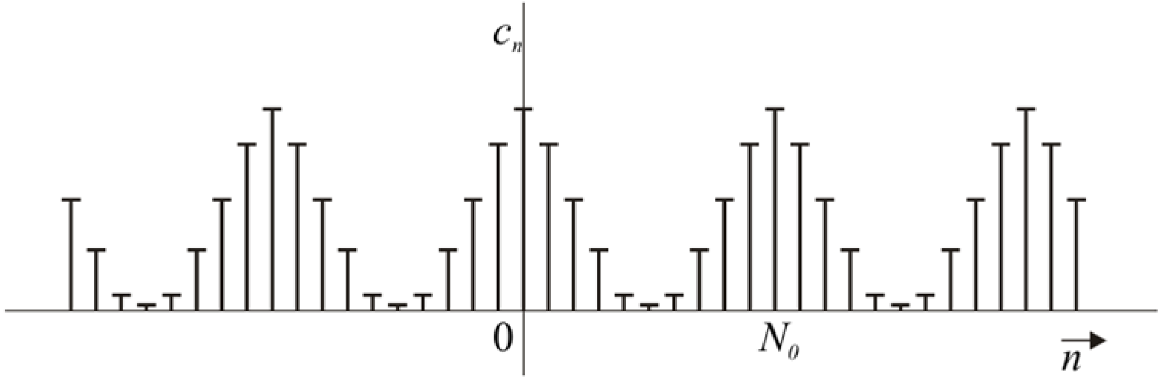
Discrete time non-periodic signal

- Decomposition into Fourier series

$$\boxed{s(k) = \sum_{n=-\infty}^{\infty} c_n e^{jnk\frac{2\pi}{N_0}} \qquad c_n = \frac{1}{N_0}\sum_{k=k_0}^{k_0+N_0-1} s(k) e^{-jnk\frac{2\pi}{N_0}}}$$

- Series of coefficients is periodical with period N_0





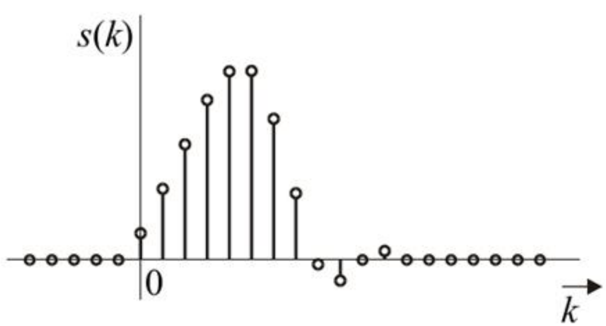
Discrete time non-periodic signal

- Discrete time Fourier transform - DtFT

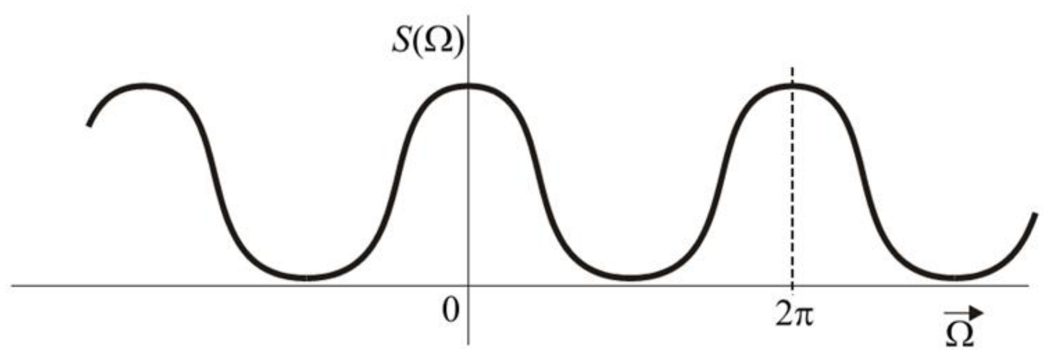
$$S(\Omega) = \mathrm{F}\left[s(k)\right] = \sum_{k=-\infty}^{\infty} s(k) e^{-j\Omega k} \qquad \text{(forward)}$$

$$s(k) = \mathrm{F}^{-1}\left[S(\Omega)\right] = \frac{1}{2\pi} \int_{(2\pi)} S(\Omega) e^{j\Omega k} d\Omega \qquad \text{(reverse)}$$

Spectrum is periodical function of frequency with period 2π



Time domain (signal)

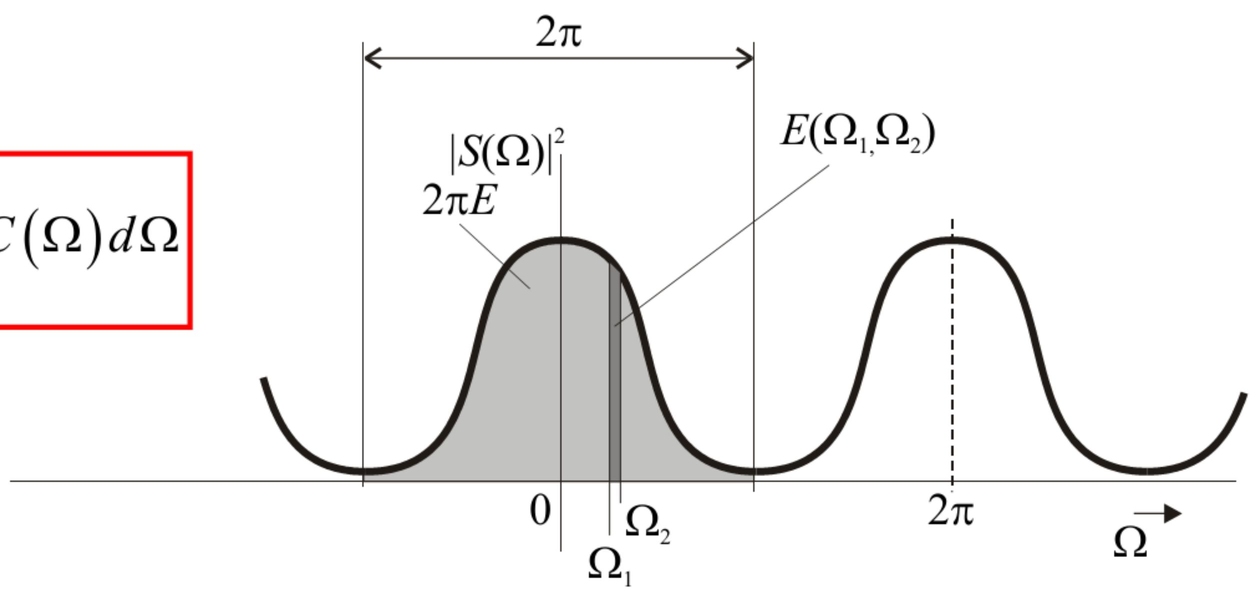


Frequency domain (spectrum)

Power spectral density

- Parseval's theorem $$E = \frac{1}{2\pi} \int_{(2\pi)} \underbrace{\left| S(\Omega) \right|^2}_{C(\Omega)} d\Omega$$
- $C(\Omega) =$ Power spectral density $$\boxed{C(\Omega) = \left| S(\Omega) \right|^2}$$
- Energy in given angle frequency interval (Ω_1, Ω_2)

$$\boxed{E = \frac{1}{2\pi} \int_{\Omega_1}^{\Omega_2} C(\Omega) d\Omega}$$



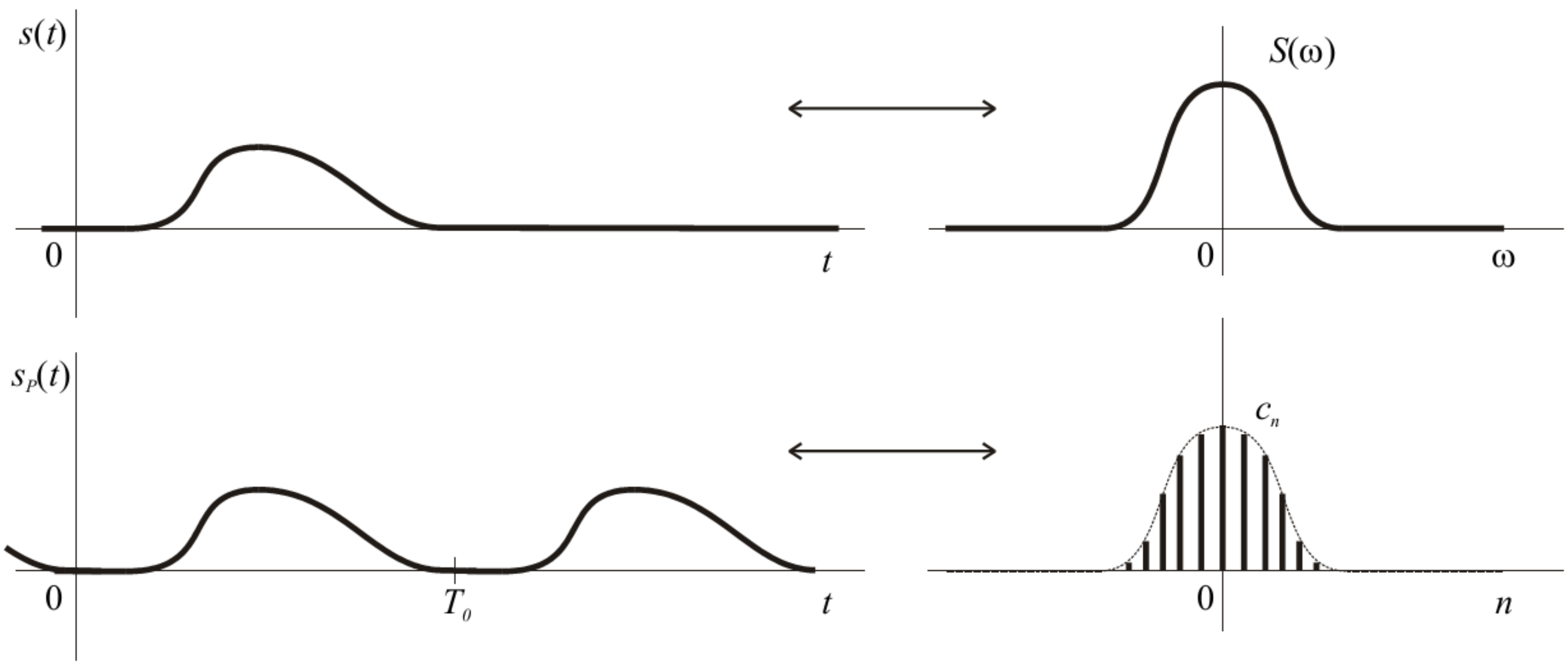
Fourier series and Fourier transform

$$s_P(t) = \sum_{m=-\infty}^{\infty} s(t - mT_0)$$

Periodic signal = Fourier series

$$s(t) = \begin{cases} s_P(t) & ; \text{ for } t \in \langle 0,{}_0 T) \\ 0 & ; \text{ Elsewhere} \end{cases}$$

Non periodic signal = Fourier transformation



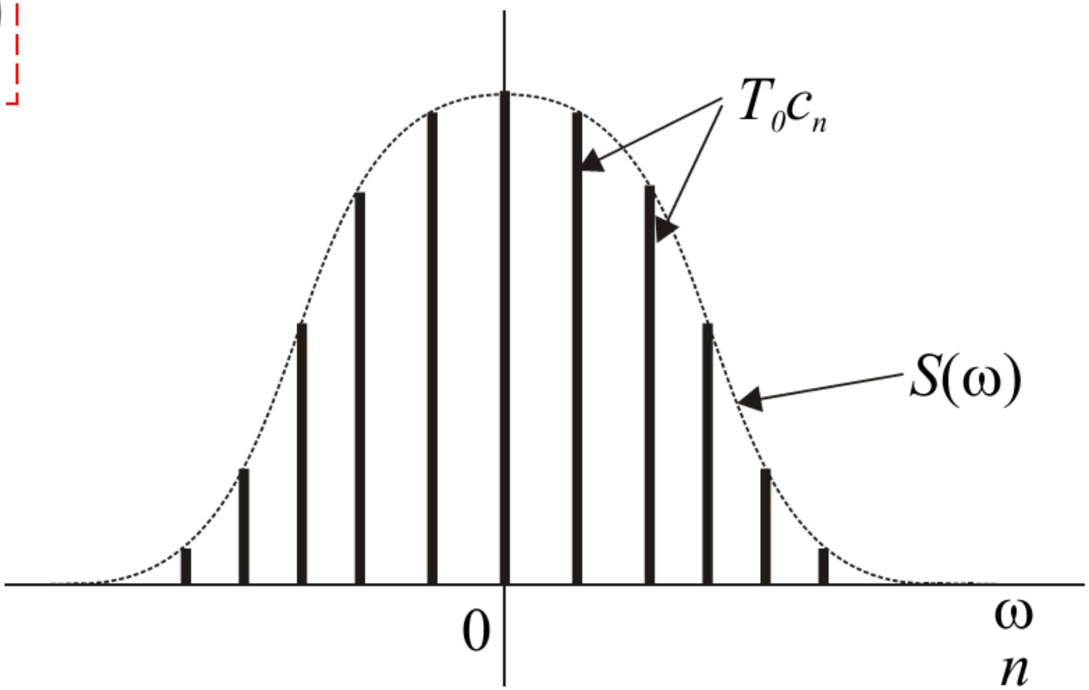
Fourier series and Fourier transform

$$s_P(t) = \sum_{m=-\infty}^{\infty} s(t - mT_0)$$

periodic signal = Fourier series

$$c_n = \frac{1}{T_0}\int_{(T_0)} s_P(t)e^{-jn\omega_0 t}\,dt = \frac{1}{T_0}\int_{t_p}^{t_p+T_0} s(t)e^{-jn\omega_0 t}\,dt = \frac{1}{T_0}\underbrace{\int_{-\infty}^{\infty} s(t)e^{-jn\omega_0 t}\,dt}_{S(n\omega_0)}$$

$$\boxed{c_n = \frac{1}{T_0}S(n\omega_0)}$$



Fourier series and Fourier transform

$$s(t)=\begin{cases} s_P(t) & ;\ \text{for } t\in\langle 0,T_0) \\ 0 & ;\ \text{Elsewhere}\end{cases}$$

Non periodic signal = Fourier transform

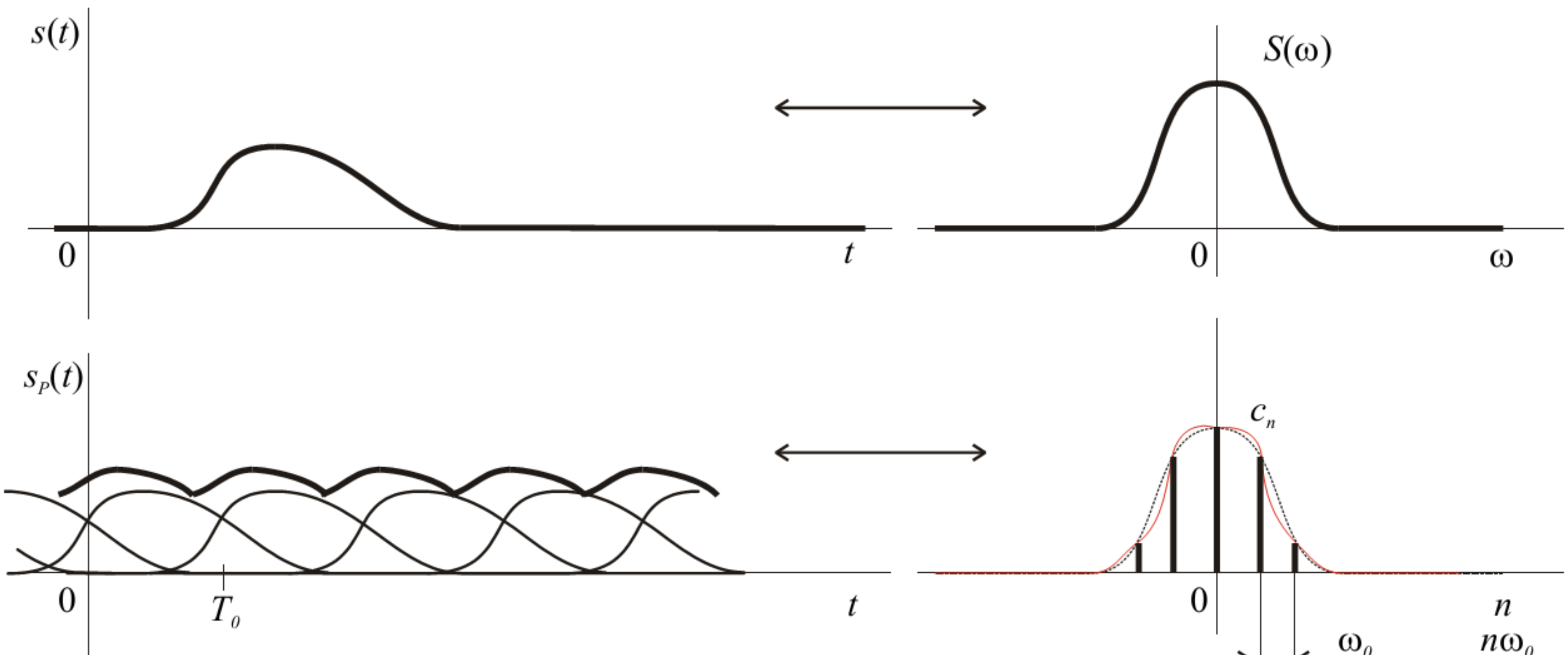
$$S(\omega)=\int_{-\infty}^{\infty} s(t)\mathrm{e}^{-j\omega t}\,dt=\int_{t_p}^{t_p+T_0} s_P(t)\mathrm{e}^{-j\omega t}\,dt=\int_{t_p}^{t_p+T_0}\sum_{n=-\infty}^{\infty} c_n\,\mathrm{e}^{jn\omega_0 t}\,\mathrm{e}^{-j\omega t}\,dt=$$

$$=\sum_{n=-\infty}^{\infty} c_n\int_{t_p}^{t_p+T_0}\mathrm{e}^{-jt(\omega-n\omega_0)}\,dt=\cdots=T_0\sum_{n=-\infty}^{\infty} c_n\,\mathrm{e}^{-j\left(t_P-\frac{T_0}{2}\right)(\omega-n\omega_0)}\,\mathrm{Sa}\left((\omega-n\omega_0)\frac{T_0}{2}\right)$$

$$\boxed{S(\omega)=\underbrace{T_0}_{\text{amplitude change}}\sum_{n=-\infty}^{\infty} c_n\,\underbrace{\mathrm{e}^{-j\left(t_P-\frac{T_0}{2}\right)(\omega-n\omega_0)}}_{\text{phase change}}\,\mathrm{Sa}\left((\omega-n\omega_0)\frac{T_0}{2}\right)}$$

Fourier series and Fourier transform

- If there is an overlap of periods of a signal in the time domain (i.e. the period T_0 is too small) is, the frequency spacing between samples of spectrum ω_0 too large and it is impossible to reconstruct continuous function $S(\omega)$ from spectrum samples $S(n\omega_0)$



Resources

Based on lectures:

- X37SGS – Signály a soustavy (Vejrazka) http://radio.feld.cvut.cz/courses/X37SGS
- ECE222 Signal fundamentals (J McNames) http://www.coursehero.com/file/4035578/FinalSolutions/

Závěr